

ON THE SINGULARITIES OF A FREE BOUNDARY THROUGH FOURIER EXPANSION

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ABSTRACT. In this paper we are concerned with singular points of solutions to the *unstable* free boundary problem

$$\Delta u = -\chi_{\{u>0\}} \quad \text{in } B_1.$$

The problem arises in applications such as solid combustion, composite membranes, climatology and fluid dynamics.

It is known that solutions to the above problem may exhibit singularities — that is points at which the second derivatives of the solution are unbounded — as well as degenerate points. This causes breakdown of by-now classical techniques. Here we introduce new ideas based on Fourier expansion of the non-linearity $\chi_{\{u>0\}}$.

The method turns out to have enough momentum to accomplish a complete description of the structure of the singular set in \mathbb{R}^3 .

A surprising fact in \mathbb{R}^3 is that although

$$\frac{u(r\mathbf{x})}{\sup_{B_1} |u(r\mathbf{x})|}$$

can converge at singularities to each of the harmonic polynomials

$$xy, \frac{x^2 + y^2}{2} - z^2 \text{ and } z^2 - \frac{x^2 + y^2}{2},$$

it may *not* converge to any of the non-axially-symmetric harmonic polynomials $\alpha((1+\delta)x^2 + (1-\delta)y^2 - 2z^2)$ with $\delta \neq 1/2$.

We also prove the existence of stable singularities in \mathbb{R}^3 .

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1. INTRODUCTION

We investigate the singular points of solutions of the unstable free boundary problem

$$(1.1) \quad \Delta u = -\chi_{\{u>0\}} \text{ in } B_1,$$

arising in solid combustion (see the references in [15]), the composite membrane problem ([9], [8], [4], [19], [10], [11]), climatology ([12]) and fluid dynamics ([1]).

The minus sign on the right-hand side drastically changes the problem from the well-known obstacle problem (see for example [5], [7] and [6]) into an unstable problem exhibiting non-uniqueness, bifurcations, unbounded second derivatives and more. Let us describe some of the known results.

From standard elliptic regularity theory it follows that if u is a solution to (1.1) then $u \in C^{1,\alpha}$ for all $\alpha < 1$. However, in contrast to the well-known obstacle problem $\Delta u = \chi_{\{u>0\}}$, the solutions to (1.1) are not $C^{1,1}$ in general. The existence of non-regular solutions was first shown in [3].

For convenience let us denote the set of singular points by

$$S^u = \{\mathbf{x} : u \notin C^{1,1}(B_r(\mathbf{x})) \text{ for any } r > 0\}.$$

As observed in [15], $\{u = 0\}$ is analytic and $u \in C^{1,1}$ in a neighborhood of each $\mathbf{x} \in \{u = 0\} \cap \{\nabla u \neq 0\}$. Thus the set of singular points is contained in the set where both u and $|\nabla u|$ vanish.

We may expect that for $\mathbf{x}^0 \in S^u$ the blow-up

$$(1.2) \quad \lim_{r \rightarrow 0} \frac{u(r\mathbf{x} + \mathbf{x}^0)}{\sup_{B_r(\mathbf{x}^0)} |u|},$$

should give us some information about the singular set. It was shown in [15] (see also Proposition 3.2 below) that at a singular point \mathbf{x}^0

$$\lim_{j \rightarrow \infty} \frac{u(r_j \mathbf{x} + \mathbf{x}^0)}{\sup_{B_{r_j}(\mathbf{x}^0)} |u|} = p(\mathbf{x}),$$

where p is a second order homogeneous harmonic polynomial. This raises several questions.

- (i) Does p depend on the choice of the sequence $r_j \rightarrow 0$?
- (ii) Does every second order homogeneous harmonic polynomial p appear as limit?
- (iii) Is there any partial regularity of the singular set?
- (iv) Do energy minimising singularities exist?

Concerning uniqueness of blow-up limits it has been shown in [15] that in two dimensions the free boundary of the *minimal solution* close to points where the second derivative is unbounded, consists of four Lipschitz graphs meeting at right angles. In [2] this fact has been extended to any solution in two dimensions, proving also uniformity and quantitative estimates by methods closely related to those in the present paper. Concerning question (iv) it has been proved in [15] that the singularity in two dimensions is *unstable* in the sense that the second variation of the energy is negative. As to stability of higher dimensional singularities there is a gap in the proof of [15] which has been pointed out by Carlos Kenig-Sagun Chanillo-Tung To ([11]). In the present paper we will prove the following main results which among other things close the gap in [15] by showing that the cut-off dimension concerning this problem is 3, that is, there exist stable singularities in three dimensions.

Main results:

- (i) *Existence of a true three-dimensional singularity (Corollary 6.1).*
- (ii) *Axial symmetry of blow-up limits in three dimensions (Theorem 8.1).*
- (iii) *Unique tangent cones at true three-dimensional singularities in \mathbb{R}^3 (Theorem 9.1).*
- (iv) *Unique tangent cones at unstable codimension two singularities in \mathbb{R}^3 (Theorem 11.1).*
- (v) *Stability of true three-dimensional singularities in \mathbb{R}^3 (Theorem 10.1).*
- (vi) *Regularity of the singular set in three dimensions (Section 12).*

Discussion. *In contrast to the analysis of singularities for minimisers or stable solutions, where there are many methods available, there are few results on unique tangent cones at unstable singularities. Even the Łojasiewicz inequality approach (see for example [20]) would be hard to realize in our problem due to the lack of a suitable local Lyapunov functional; we do have a monotonicity formula playing the role of a local Lyapunov functional, but as it turns out it has the wrong scaling to be used at the unstable singularities of “supercharacteristic growth”.*

The natural approach would be to study blow-up limits in order to analyze the singularities. Unfortunately the blow-up sequence in (1.2) does not provide enough information of the solution as the nonlinearity of equation (1.1) vanishes in the limit. To preserve some information of the nonlinearity we will instead, in Section 3, consider

$$(1.3) \quad \frac{u(r_j \mathbf{x} + \mathbf{x}^0)}{r_j^2} - \Pi(u, r_j, \mathbf{x}^0),$$

where $\Pi(u, r_j, \mathbf{x}^0)$ is the projection of $u(r_j \mathbf{x} + \mathbf{x}^0)/r_j^2$ in B_1 onto the homogeneous harmonic second order polynomials (see Definition 3.5).

It can be shown that if

$$\lim_{j \rightarrow \infty} \frac{u(r_j \mathbf{x} + \mathbf{x}^0)}{\sup_{B_{r_j}(\mathbf{x}^0)} |u|} = p(\mathbf{x}),$$

then

$$\lim_{j \rightarrow \infty} \left(\frac{u(r_j \mathbf{x} + \mathbf{x}^0)}{r_j^2} - \Pi(u, r_j, \mathbf{x}^0) \right) = Z_p,$$

where Z_p is a solution of

$$\Delta Z_p = -\chi_{\{p>0\}}.$$

Next we notice that, at each singular point \mathbf{x}^0 ,

$$\lim_{j \rightarrow \infty} \frac{\Pi(u, r_j, \mathbf{x}^0)}{\sup_{B_1} |\Pi(u, r_j, \mathbf{x}^0)|} = \lim_{j \rightarrow \infty} \frac{u(r_j \mathbf{x} + \mathbf{x}^0)}{\sup_{B_{r_j}(\mathbf{x}^0)} |u|}.$$

So in order to prove uniqueness of p it is sufficient to control how $\Pi(u, r, \mathbf{x}^0)$ changes when r varies. More precisely we would want to estimate

$$(1.4) \quad \left| \frac{\Pi(u, r, \mathbf{x}^0)}{\sup_{B_1} |\Pi(u, r, \mathbf{x}^0)|} - \frac{\Pi(u, r/2, \mathbf{x}^0)}{\sup_{B_1} |\Pi(u, r/2, \mathbf{x}^0)|} \right|.$$

Our method of proof is based on the observation that $u(r\mathbf{x} + \mathbf{x}^0) \approx \tau_r p_r + Z_{p_r}$ in B_r , where p_r is a second order harmonic polynomial of norm 1. It follows that $\Pi(u, r/2, \mathbf{x}^0) \approx \Pi(\tau_r p_r + Z_{p_r}, 1/2, 0) = \Pi(\tau_r p_r, 1/2, 0) + \Pi(Z_{p_r}, 1/2, 0) = \tau_r p_r + \Pi(Z_{p_r}, 1/2, 0)$ (cf. Section 7). Therefore it is essential to control $\Pi(Z_{p_r}, \cdot)$ in order to estimate (1.4). This control will be achieved by means of an explicit calculation of the Fourier coefficients of Z_{p_r} .

Plan of the paper.

In Section 3 we will remind ourselves of results and definitions of [22], [15], [3] and [2] that are relevant to the present paper.

In Section 4 we use techniques developed in [14] based on Fourier coefficients to analyze Z_p . We also explicitly calculate Z_p when $p = 2xz$ or $p = \pm((x^2 + y^2)/2 - z^2)$. Using these calculations we are able to show in Section 5 that in three dimensions and for small r

$$\sup_{B_r} |u(\mathbf{x} + \mathbf{x}^0)| \geq cr^2 |\log r|$$

(Corollary 5.3).

Based on this estimate on the growth of u we prove in Section 6 existence of a true three-dimensional singularity.

Section 7 provides estimates on $u - \Pi(u) - Z_{\Pi(u)}$ which we combine in Section 8 carefully with the above analysis of Z_p to show axial symmetry of blow-up limits in three dimensions. This is a remarkable symmetrization effect in view of the fact that there are of course second order homogeneous harmonic polynomials that are *not* axially symmetric.

In Sections 9 and 11 we prove —once more carefully using the information gained on the Fourier coefficients— uniqueness of the blow-up limits at singular points in \mathbb{R}^3 . Based on the asymptotics in Section 9 we are able to show in Section 10 that true three-dimensional singularities are stable. In Section 12 we use standard techniques to show that in the three-dimensional case the singular set may be decomposed into a countable set of isolated points and a component that is locally contained in a C^1 -curve. In the Appendix we have gathered technical calculations which may well be considered to be the core of our paper.

2. NOTATION

Throughout this article \mathbb{R}^n will be equipped with the Euclidean inner product $\mathbf{x} \cdot \mathbf{y}$ and the induced norm $|\mathbf{x}|$. Moreover $A : B = \sum_{i,j=1}^n a_{ij} b_{ij}$ shall denote the inner product of two (n, n) matrices. We will use the set \mathcal{Q} of all orthogonal matrices in \mathbb{R}^n . We define $B_r(\mathbf{x}^0)$ as the open n -dimensional ball of center \mathbf{x}^0 ,

radius r and volume $r^n \omega_n$, and $B_r^+(\mathbf{x}^0) := \{\mathbf{x} \in B_r(\mathbf{x}^0) : x_n > 0\}$, $B_r^-(\mathbf{x}^0) := \{\mathbf{x} \in B_r(\mathbf{x}^0) : x_n < 0\}$. When not specified, \mathbf{x}^0 is assumed to be 0. We shall often use abbreviations for inverse images like $\{u > 0\} := \{\mathbf{x} \in \Omega : u(\mathbf{x}) > 0\}$, $\{x_n > 0\} := \{\mathbf{x} \in \mathbb{R}^n : x_n > 0\}$ etc. and occasionally we shall employ the decomposition $\mathbf{x} = (x_1, \dots, x_n)$ of a vector $\mathbf{x} \in \mathbb{R}^n$. We will use the n -dimensional Lebesgue measure \mathcal{L}^n and the k -dimensional Hausdorff measure \mathcal{H}^k . When considering a set A , χ_A shall stand for the characteristic function of A , while ν shall typically denote the outward normal to a given boundary. We will use Landau's symbols as signed variables. For example $-o(1)$ will mean a negative quantity that turns to zero. By \mathbb{P}_2 we will denote the space of second order homogeneous harmonic polynomials in \mathbb{R}^n . We shall also use the projection Π onto \mathbb{P}_2 as well as the norm τ of $\Pi(v)$, both defined in Definition 3.5, as well as the parametrization parameters $\delta^A(v)$ and $\delta^B(v)$ defined in Definition 4.2. Last, we shall use for $p \in \mathbb{P}_2$ the Newtonian potential Z_p , i.e. the unique solution of

$$\begin{aligned} \Delta Z_p &= -\chi_{\{p>0\}} \text{ in } \mathbb{R}^n, \\ Z_p(0) &= |\nabla Z_p(0)| = 0, \\ \lim_{|\mathbf{x}| \rightarrow \infty} \frac{Z(\mathbf{x})}{|\mathbf{x}|^3} &= 0 \text{ and} \\ \Pi(Z_p, 1) &= 0 \text{ (cf. [2] Section 4).} \end{aligned}$$

3. GENERAL BACKGROUND

In this section we will gather some results from [15] and [3], and describe some compactness properties of blow-ups of solutions. First we will remind ourselves of the monotonicity formula proved in [22]. The roots of those monotonicity formulas are harmonic mappings ([18], [17]) and blow-up ([16]).

Theorem 3.1. *Suppose that u is a solution of (1.1) in Ω and that $B_\delta(\mathbf{x}^0) \subset \Omega$. Then for all $0 < \rho < \sigma < \delta$ the function*

$$\begin{aligned} \Phi_{\mathbf{x}^0}^u(r) &:= r^{-n-2} \int_{B_r(\mathbf{x}^0)} \left(|\nabla u|^2 - 2 \max(u, 0) \right) \\ &\quad - 2 r^{-n-3} \int_{\partial B_r(\mathbf{x}^0)} u^2 d\mathcal{H}^{n-1}, \end{aligned}$$

defined in $(0, \delta)$, satisfies the monotonicity formula

$$\Phi_{\mathbf{x}^0}^u(\sigma) - \Phi_{\mathbf{x}^0}^u(\rho) = \int_\rho^\sigma r^{-n-2} \int_{\partial B_r(\mathbf{x}^0)} 2 \left(\nabla u \cdot \nu - 2 \frac{u}{r} \right)^2 d\mathcal{H}^{n-1} dr \geq 0.$$

This energy monotonicity is important since it helps us to distinguish different points of the set $\{u(0) = |\nabla u(0)| = 0\}$. In particular we may according to the following Proposition define the singular set S^u as

$$S^u = \{\mathbf{x} \in B_1 : u(\mathbf{x}) = |\nabla u(\mathbf{x})| = 0 \text{ and } \lim_{r \rightarrow 0} \Phi_{\mathbf{x}}^u(r) = -\infty\}.$$

Proposition 3.2 (Proposition 5.1 in [15]). *Let u be a solution of (1.1) in Ω and let us consider a point $\mathbf{x}^0 \in \Omega \cap \{u = 0\} \cap \{\nabla u = 0\}$.*

(i) *In the case $\Phi_{\mathbf{x}^0}^u(0+) = -\infty$, $\lim_{r \rightarrow 0} r^{-3-n} \int_{\partial B_r(\mathbf{x}^0)} u^2 d\mathcal{H}^{n-1} = +\infty$, and for*

$$T(\mathbf{x}^0, r) := \left(r^{1-n} \int_{\partial B_r(\mathbf{x}^0)} u^2 d\mathcal{H}^{n-1} \right)^{\frac{1}{2}}, \text{ each limit of } \frac{u(\mathbf{x}^0 + r\mathbf{x})}{T(\mathbf{x}^0, r)}$$

as $r \rightarrow 0$ belongs to \mathbb{P}_2 .

(ii) In the case $\Phi_{\mathbf{x}^0}^u(0+) \in (-\infty, 0)$,

$$u_r(\mathbf{x}) := \frac{u(\mathbf{x}^0 + r\mathbf{x})}{r^2}$$

is bounded in $W^{1,2}(B_1(0))$, and each limit as $r \rightarrow 0$ is a homogeneous solution of degree 2.

(iii) If $\Phi_{\mathbf{x}^0}^u(0+) = 0$, then

$$\frac{u(\mathbf{x}^0 + r\mathbf{x})}{r^2} \rightarrow 0 \text{ in } W^{1,2}(B_1(0)) \text{ as } r \rightarrow 0.$$

Remark 3.3. In [15, Lemma 5.2] it says that case (ii) in Proposition 3.2 does not occur in \mathbb{R}^2 . Unfortunately the authors omitted the following homogeneous solution of second order in \mathbb{R}^2 :

Using polar coordinates (r, ϕ) , let

$$u(r, \phi) = \begin{cases} -\frac{r^2}{2} \cos^2(\phi) + \frac{r^2}{4\sqrt{3}} \sin(2\phi) & \text{when } \phi \in (\pi/3, \pi/2), \\ -\frac{r^2}{8} \cos(2\phi) - \frac{r^2}{8\sqrt{3}} \sin(2\phi) & \text{when } \phi \in (-\pi/6, \pi/3), \\ -\frac{r^2}{2} \cos^2(\phi + 2\pi/3) + \frac{r^2}{4\sqrt{3}} \sin(2\phi + 2\pi/3) & \text{when } \phi \in (-\pi/3, -\pi/6), \\ -\frac{r^2}{8} \cos(2\phi + 2\pi/3) - \frac{r^2}{8\sqrt{3}} \sin(2\phi + 2\pi/3) & \text{when } \phi \in (-5\pi/6, -\pi/3), \\ -\frac{r^2}{2} \cos^2(\phi + 4\pi/3) + \frac{r^2}{4\sqrt{3}} \sin(2\phi + 4\pi/3) & \text{when } \phi \in (-\pi, -5\pi/6), \\ -\frac{r^2}{8} \cos(2\phi + 4\pi/3) - \frac{r^2}{8\sqrt{3}} \sin(2\phi + 4\pi/3) & \text{when } \phi \in (-\pi, -3\pi/2). \end{cases}$$

Then u is a second order homogeneous solution to equation (1.1).

Let us show that up to rotations, u is the unique non-trivial second order homogeneous solution to equation (1.1) in \mathbb{R}^2 . Each cone in which u is negative has to have an opening of exactly $\pi/2$. Thus u can be negative in at most 4 different connected components. However if u is negative in four components then $u \leq 0$ and thus $\Delta u = 0$. Since $u(0) = 0$, $u \equiv 0$ by the strong maximum principle. If u is negative in only one component then $\Delta u = -1$ in a cone with opening $3\pi/2$ with zero boundary values on that cone, and it is easy to see that such a u is not homogeneous of second order. If $u \leq 0$ in two components then, after a rotation, $\Delta u = -\chi_{\{xy > 0\}}$. In “2-dimensional Solutions” (vi) in Section 4, we will see that such a solution is not homogeneous either. The only remaining possibility is that $u \leq 0$ in three components, with three components of $u > 0$ in between. As the gradient of u is continuous across the zero level set and u is symmetric in each cone where u has a sign it follows that the opening of the cones where $u > 0$ must equal each other. It follows that $u > 0$ in three cones of opening $\pi/6$ where $\Delta u = -1$. Thus u is unique up to a rotation.

In [3] the authors have obtained existence of solutions in two dimensions exhibiting *cross-like singularities* at which the second derivatives of the solution are unbounded (case (i) of Proposition 3.2), as well as degenerate singularities at which the solution decays to zero faster than any quadratic polynomial (case (iii) of Proposition 3.2):

Theorem 3.4 (Cross-shaped singularity, Corollary 4.2 in [3]). *There exists a solution u of*

$$\Delta u = -\chi_{\{u>0\}} \quad \text{in } B_1 \subset \mathbb{R}^2$$

*that is **not** of class $C^{1,1}$. Each limit of*

$$\frac{u(r\mathbf{x})}{T(0, r)}$$

as $r \rightarrow 0$ coincides after rotation with the function $(x_1^2 - x_2^2)/\|x_1^2 - x_2^2\|_{L^2(\partial B_1(0))}$.

Remark on the Proof: In [3] the authors show that one can construct a solution u to (1.1) with $\lim_{r \rightarrow 0} \Phi_0^u(r) \leq -M$ for any $M \geq 0$. Then they use [15, Lemma 5.2] that states that if $\lim_{r \rightarrow 0} \Phi_0^u(r) < 0$ in \mathbb{R}^2 then we are in case (i) of Proposition 3.2. As we pointed out in Remark 3.3, Lemma 5.2 in [15] is not true. The proof in [3] however is easily fixed: we only have to notice that all second order homogeneous solutions of (1.1) have fixed energy $\Phi_0^u = m_0$ which follows from the uniqueness in Remark 3.3. Thus if we choose the constant M large enough we can exclude the possibility that we are in case (ii) or (iii) of Proposition 3.2 and the theorem follows. \square

The proof of the previous theorem can be adjusted to construct other kinds of singular points (see Corollary 6.1).

Definition 3.5. By $\Pi(u, r, \mathbf{x}^0)$ we will denote the projection operator onto \mathbb{P}_2 defined as follows: $\Pi(u, r, \mathbf{x}^0) = \tau p$, where $\tau \in \mathbb{R}^+$ and $p \in \mathbb{P}_2$ satisfies $\sup_{B_1} |p| = 1$ as well as

$$\inf_{h \in \mathbb{P}_2} \int_{B_1(0)} \left| \frac{D^2 u(r\mathbf{x} + \mathbf{x}^0)}{r^2} - D^2 h \right|^2 = \int_{B_1(0)} \left| \frac{D^2 u(r\mathbf{x} + \mathbf{x}^0)}{r^2} - \tau D^2 p \right|^2.$$

We will often write $\Pi(u, r)$ when \mathbf{x}^0 is either the origin or given by the context. At times we will also denote $\tau_r = \sup_{B_1} |\Pi(u, r)|$ and $p_r = \Pi(u, r)/\tau_r$.

The following Lemma justifies the previous Definition.

Lemma 3.6. *The following four statements hold.*

- (i) *For each $v \in W^{2,2}(B_1)$ the minimizer of Definition 3.5 exists and is unique. Thus $\Pi : W^{2,2}(B_1) \times (0, s) \times B_{1-s} \rightarrow P$ is well-defined.*
- (ii) *Π is a linear operator.*
- (iii) *If $h \in W^{2,2}(B_1)$ is harmonic in B_1 then $\Pi(h(\mathbf{x}), s) = \Pi(h(\mathbf{x}), r)$ for all $s, r \in (0, 1)$.*
- (iv) *For every $v, w \in W^{2,2}(B_1)$,*

$$\sup_{B_1} |\Pi(v + w, r)| \leq \sup_{B_1} |\Pi(v, r)| + \sup_{B_1} |\Pi(w, r)|.$$

Proof. The first and second statement follow from the projection theorem with respect to the $L^2(B_1; \mathbb{R}^{n^2})$ -inner product and the linear subspace $\{f \in L^2(B_1; \mathbb{R}^{n^2}) : f(\mathbf{x}) \text{ is symmetric, constant and } \text{trace}(f(\mathbf{x})) = 0\}$.

Writing h as the sum of homogeneous harmonic polynomials h_j that are orthogonal to each other with respect to

$$(v, w) := \int_{B_1} \sum_{i,j=1}^n \partial_{ij} v \partial_{ij} w,$$

we see that $\Pi(h_j) = 0$ for all j such that the degree of h_j is different from 2, implying the third statement.

The last statement follows from the linearity of Π and the triangle inequality in $L^2(B_1; \mathbb{R}^{n^2})$. \square

Next we mention that solutions to (1.1) have second derivatives in BMO . This has been proved in [2, Lemma 5.1] using standard facts of harmonic analysis.

Proposition 3.7 (cf. [2, Lemma 5.1]). *Let u be a solution to (1.1) in B_1 such that $\sup_{B_1} |u| \leq M$ and $u(0) = |\nabla u(0)| = 0$. Then*

$$\sup_{B_1} \left| \frac{u(r\mathbf{x})}{r^2} - \Pi(u, r) \right| \leq C_0 \text{ for every } r \leq \frac{1}{2}$$

where the constant C_0 depends only on M and n .

Furthermore, for each $\alpha < 1$

$$\left\| \frac{u(r\mathbf{x})}{r^2} - \Pi(u, r) \right\|_{C^{1,\alpha}(\overline{B_1})} \leq C_\alpha(M, n),$$

and for each $p < \infty$

$$\left\| \frac{u(r\mathbf{x})}{r^2} - \Pi(u, r) \right\|_{W^{2,p}(B_1)} \leq C_p(M, n).$$

4. FOURIER SERIES EXPANSIONS OF GLOBAL SOLUTIONS

In this section we will remind ourselves of the work of L. Karp and A.S. Margulis [14]. In particular, Theorem 3.1 and Proposition 3.2 in [14], summarized in the next theorem, will be of importance to us.

Theorem 4.1 ([14]). *Let $\sigma \in L^\infty(\mathbb{R}^n)$ be homogeneous of zeroth order, that is $\sigma(\mathbf{x}) = \sigma(r\mathbf{x})$ for all $r > 0$. Assume that σ has the Fourier series expansion*

$$\sigma(\mathbf{x}) = \sum_{i=0}^{\infty} a_i \sigma_i,$$

on the unit sphere, where σ_i is a homogeneous harmonic polynomial of order i .

Moreover assume that $\Delta u = \sigma$ and that $u(0) = |\nabla u(0)| = \lim_{\mathbf{x} \rightarrow \infty} u(\mathbf{x})/|\mathbf{x}|^3 = 0$. Then

$$u(\mathbf{x}) = q(\mathbf{x}) \log |\mathbf{x}| + |\mathbf{x}|^2 \phi(\mathbf{x}),$$

where

$$q = \frac{a_2}{n+2} \sigma_2$$

and

$$\phi(\mathbf{x}) = |\mathbf{x}|^2 \sum_{i \neq 2} \frac{a_i}{(n+i)(i-2)} \sigma_i\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right).$$

Let us explain how we are going to use Theorem 4.1 in the present paper: If u is a solution to equation (1.1) such that $u(0) = |\nabla u(0)| = 0$, if

$$\lim_{j \rightarrow \infty} \frac{u(r_j \mathbf{x})}{\sup_{B_{r_j}} |u|} = p$$

for some $p \in \mathbb{P}_2$ and some sequence $r_j \rightarrow 0$, and if

$$\lim_{j \rightarrow \infty} \left(\frac{u(r_j \mathbf{x})}{r_j^2} - \Pi(u, r_j) \right) = Z_p,$$

then by $C^{1,\alpha}$ -convergence we will have $Z_p(0) = |\nabla Z_p(0)| = 0$ as well as $\Delta Z_p = -\chi_{\{p>0\}}$. Also, by weak $W^{2,2}$ -convergence we will have

$$0 = \int_{B_1} D^2 Z_p : D^2 h$$

for all $h \in \mathbb{P}_2$. The latter is equivalent to $\Pi(Z_p, 1) = 0$.

By Theorem 4.1 with $\sigma = -\chi_{\{p>0\}}$ we can write

$$Z_p(\mathbf{x}) = q(\mathbf{x}) \log |\mathbf{x}| + |\mathbf{x}|^2 \phi(\mathbf{x}),$$

where $q = \frac{a_2}{n+2} \sigma_2$. From here on we will assume that $n = 3$ and $\mathbf{x} = (x, y, z)$. It will also be convenient to parametrize the second order harmonic polynomials. We will assume that $p = p_\delta := (1/2 + \delta)x^2 + (1/2 - \delta)y^2 - z^2$. This can be done without loss of generality since there is always a rotation of the coordinate system such that $D^2 p$ is a diagonal matrix. Rotating the coordinate system in that way, and if necessary renaming x , y and z we can always make sure that p is up to a scaling factor of the form above or that $-p = (1/2 + \delta)x^2 + (1/2 - \delta)y^2 - z^2$. The latter case can be handled similarly. We would want to calculate σ_2 . To that end we choose the polynomials $3x^2 - |\mathbf{x}|^2$, $3y^2 - |\mathbf{x}|^2$ and $3z^2 - |\mathbf{x}|^2$ spanning the axisymmetric second order harmonic polynomials in \mathbb{R}^3 . That choice is somewhat arbitrary, but we contend that choosing different polynomials would not facilitate substantially anything that follows. It follows that

$$\sigma_2 = C((3A_x(\delta) - A(\delta))x^2 + (3A_y(\delta) - A(\delta))y^2 + (3A_z(\delta) - A(\delta))z^2),$$

where C has been chosen such that $\|\sigma\|_{L^2(\partial B_1)} = 1$.

Using spherical coordinates $x = r \sin(\theta) \cos(\phi)$, $y = r \sin(\theta) \sin(\phi)$ and $z = r \cos(\theta)$, the characteristic function $\chi_{\{p_\delta>0\}} = \chi_{\{\theta > \arccot(\sqrt{1/2+\delta \cos(2\phi)})\}}$. The coefficients satisfy

$$A_x(\delta) = \int_{\partial B_1} -\chi_{\{p_\delta>0\}} x^2 = -8 \int_0^{\pi/2} \int_{\arccot(\sqrt{1/2+\delta \cos(2\phi)})}^{\pi/2} \sin^3(\theta) \cos^2(\phi) d\theta d\phi,$$

$$A_y(\delta) = \int_{\partial B_1} -\chi_{\{p_\delta>0\}} y^2 = -8 \int_0^{\pi/2} \int_{\arccot(\sqrt{1/2+\delta \cos(2\phi)})}^{\pi/2} \sin^3(\theta) \sin^2(\phi) d\theta d\phi,$$

$$A_z(\delta) = \int_{\partial B_1} -\chi_{\{p_\delta>0\}} z^2 = -8 \int_0^{\pi/2} \int_{\arccot(\sqrt{1/2+\delta \cos(2\phi)})}^{\pi/2} \sin(\theta) \cos^2(\theta) d\theta d\phi$$

and

$$A(\delta) = \int_{\partial B_1} -\chi_{\{p_\delta>0\}} = -8 \int_0^{\pi/2} \int_{\arccot(\sqrt{1/2+\delta \cos(2\phi)})}^{\pi/2} \sin(\theta) d\theta d\phi.$$

Next we notice that with

$$\mathbf{K}_0 = \frac{2 \log 2}{5 \|3x^2 - 1\|_{L^2(\partial B_1)}^2},$$

(4.1)

$$\Pi(Z_{p_\delta}, 1/2) = -\mathbf{K}_0((3A_x(\delta) - A(\delta))x^2 + (3A_y(\delta) - A(\delta))y^2 + (3A_z(\delta) - A(\delta))z^2),$$

since $\Pi(\sigma_i) = 0$ for $i \neq 2$. Calculating A_x , A_y , A_z and A we may estimate the rotation of $\Pi(u, r)$ as follows: If $u(r\mathbf{x})/r^2 - \Pi(u, r) \approx Z_{p_\delta}$ then

$$\Pi(u, r) - \Pi(u, r/2) \approx -\Pi(Z_{p_\delta}, 1/2).$$

Later on this will be our main tool to analyze singular points.

For convenience we will later also use the alternative representation $p_\delta = (1 - \delta)x^2 + \delta y^2 - z^2$ leading to the coefficients

$$B_x(\delta) = -8 \int_0^{\pi/2} \int_{\arccot(\sqrt{(1-\delta)\cos^2(\phi)+\delta\sin^2(\phi)})}^{\pi/2} \sin^3(\theta) \cos(\phi)^2 d\theta d\phi,$$

$$B_y(\delta) = -8 \int_0^{\pi/2} \int_{\arccot(\sqrt{(1-\delta)\cos^2(\phi)+\delta\sin^2(\phi)})}^{\pi/2} \sin^3(\theta) \sin(\phi)^2 d\theta d\phi$$

and

$$B(\delta) = -8 \int_0^{\pi/2} \int_{\arccot(\sqrt{(1-\delta)\cos^2(\phi)+\delta\sin^2(\phi)})}^{\pi/2} \sin(\theta) d\theta d\phi.$$

That is, $B_x(1/2 - \delta) = A_x(\delta)$ etc. It will be convenient to define the parameter δ for polynomials and solutions:

Definition 4.2. For each $v \in W^{2,2}(B_1)$, let, if necessary after rotation,

$$\Pi(v, 1)/\sup_{B_1} |\Pi(v, 1)| = (1/2 + \delta)x^2 + (1/2 - \delta)y^2 - z^2$$

$$\text{or } -(1/2 + \delta)x^2 - (1/2 - \delta)y^2 + z^2.$$

We note that δ is unique and define $\delta^A(v) := \delta$.

Moreover, let

$$\Pi(v, 1)/|\sup_{B_1} \Pi(v, 1)| = (1 - \tilde{\delta})x^2 + \tilde{\delta}y^2 - z^2$$

$$\text{or } -(1 - \tilde{\delta})x^2 - \tilde{\delta}y^2 + z^2.$$

We note that $\tilde{\delta}$ is unique and define $\delta^B(v) := \tilde{\delta}$. It is important to note that $\sup_{B_1} |\Pi(v, 1)|/\sup_{B_1} |\Pi(v, 1)| = 1$ implies $\delta^B(v) = \tilde{\delta} \geq 0$.

We will also use $\delta_r^A := \delta^A(u(r \cdot))$ and $\delta_r^B := \delta^B(u(r \cdot))$.

In general we cannot calculate the integrals A_x, A_y, \dots explicitly. In some special cases however, when we have sufficient symmetry, we may even write down explicit solutions to the equation $\Delta u = -\chi_{\{p>0\}}$. Luckily and surprisingly, as seen in Section 8, these special solutions are the only solutions appearing as limits of

$$\frac{u(r\mathbf{x} + \mathbf{x}^0)}{r^2} - \Pi(u, r, \mathbf{x}^0).$$

1. 2-dimensional Solutions (cf. [2, Lemma 4.4]):

Define $v : (0, +\infty) \times [0, +\infty) \rightarrow \mathbb{R}$ by

$$v(x, z) := -4xz \log(x^2 + z^2) + 2(x^2 - z^2) \left(\frac{\pi}{2} - 2 \arctan\left(\frac{z}{x}\right) \right) - \pi(x^2 + z^2).$$

Moreover, let

$$w(x, z) := \begin{cases} v(x, z), & xz \geq 0, x \neq 0, \\ -v(-x, z), & x < 0, z \geq 0, \\ -v(x, z), & x > 0, z \leq 0, \end{cases}$$

and define

$$Z(x, z) := \frac{w(x, z) - \pi(x^2 + z^2) + 8xz}{8\pi}.$$

It has been shown in [2, Lemma 4.4] that

- (i) $\Delta Z = -\chi_{\{xz > 0\}}$ in \mathbb{R}^3 .
- (ii) $Z(0) = |\nabla Z(0)| = 0$.
- (iii) $\lim_{|\mathbf{x}| \rightarrow \infty} \frac{Z(\mathbf{x})}{|\mathbf{x}|^3} = 0$.
- (iv) $\Pi(Z, 1) = 0$.
- (v) $\Pi(Z, 1/2) = \log(2)xz/\pi$, $\tau(Z, 1/2) = \log(2)/(2\pi)$.
- (vi) Z is the unique function satisfying (i)-(iv).

2. True 3D Solutions:

Next we are going to calculate Z_p for $p(\mathbf{x}) = (x^2 + y^2)/2 - z^2$.

Let us denote

$$v_1 = 2p(\mathbf{x}) \log(x^2 + y^2) - 4z^2$$

and

$$v_2 = -\frac{3z|\mathbf{x}|}{2} + \frac{1}{2}p(\mathbf{x}) \log\left(\frac{|\mathbf{x}| - z}{|\mathbf{x}| + z}\right).$$

Then $\Delta v_1 = 0$ and $\Delta v_2 = 0$ in $\mathbb{R}_+^3 \setminus \{x^2 + y^2 = 0\}$. Also notice that $\partial_z v_1(x, y, 0) = 0$.

Let

$$v(\mathbf{x}) = \frac{\sqrt{3}}{36} (4v_2(\mathbf{x}) - v_1(\mathbf{x}))$$

in $\{p(\mathbf{x}) \leq 0\} \cap \{z > 0\}$, where the coefficients for v_1 and v_2 are chosen such that the singularities cancel at $x = y = 0$. Moreover, let

$$\begin{aligned} v(\mathbf{x}) = & -\frac{\sqrt{3}}{36} v_1(\mathbf{x}) \\ & + \frac{3 + \sqrt{3} \log(2 - \sqrt{3})}{18} p(\mathbf{x}) - \frac{|\mathbf{x}|}{6} \end{aligned}$$

in $\{p(\mathbf{x}) > 0\} \cap \{z > 0\}$.

Next we reflect v at $\{z = 0\}$ according to

$$\tilde{Z}_1(\mathbf{x}) = \begin{cases} v(x, y, z) & \text{if } z \geq 0, \\ v(x, y, -z) & \text{if } z < 0. \end{cases}$$

Last, we define $Z_1 = \tilde{Z}_1 - \Pi(\tilde{Z}_1, 1)$ and $Z_2 = -Z_1$.

We have thus established the following lemma:

Lemma 4.3. *Let Z , Z_1 and Z_2 be as above. Then, with $p(\mathbf{x}) = (x^2 + y^2)/2 - z^2$,*

- (i) $\Delta Z = -\chi_{\{xz > 0\}}$, $Z(0) = |\nabla Z(0)| = 0$, $\lim_{|\mathbf{x}| \rightarrow \infty} |Z(\mathbf{x})|/|\mathbf{x}|^3 = 0$ and $\Pi(Z, 1) = 0$, $\Pi(Z, 1/2) = (\log(2)/\pi)xz$.
- (ii) $\Delta Z_1 = -\chi_{\{p(\mathbf{x}) > 0\}}$, $Z_1(0) = |\nabla Z_1(0)| = 0$, $\lim_{|\mathbf{x}| \rightarrow \infty} |Z_1(\mathbf{x})|/|\mathbf{x}|^3 = 0$ and $\Pi(Z_1, 1) = 0$, $\Pi(Z_1, 1/2) = \log(2)(\sqrt{3}/9)p(\mathbf{x})$.

- (iii) $\Delta Z_2 = -\chi_{\{p(\mathbf{x}) < 0\}}$, $Z_2(0) = |\nabla Z_2(0)| = 0$,
 $\lim_{\mathbf{x} \rightarrow \infty} |Z_2(\mathbf{x})|/|\mathbf{x}|^3 = 0$
and $\Pi(Z_2, 1) = 0$, $\Pi(Z_2, 1/2) = -\log(2)(\sqrt{3}/9)p(\mathbf{x})$.

Proof. The proof follows from simple calculation. \square

Remark 4.4. The fact that $\Pi(Z_p, 1/2)$ is a multiple of the polynomial p in the above three cases, natural though it may be, will be of paramount importance in later chapters when it comes to the question of unique tangent cones.

The following two collections of properties of the A's and B's visualized in Figure 1-5 are of central importance in our paper and will be proved in the Appendix together with Lemma 4.6 below.

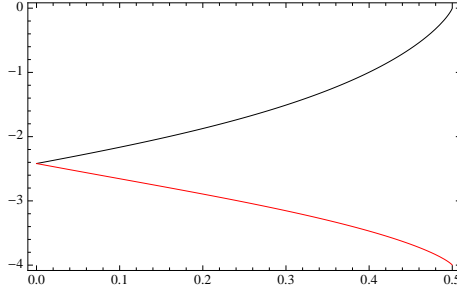


FIGURE 1. $3A_y(\delta) - A(\delta)$ (upper graph) versus $3A_x(\delta) - A(\delta)$

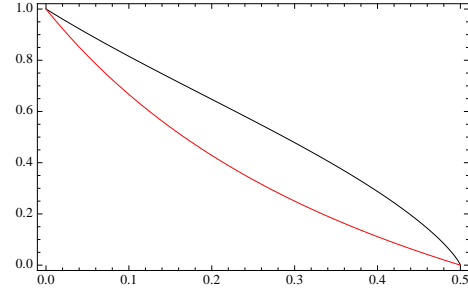


FIGURE 2. $\frac{3A_y(\delta) - A(\delta)}{3A_x(\delta) - A(\delta)}$ (upper graph) versus $\frac{1-2\delta}{1+2\delta}$

Theorem 4.5. For $\delta \in (0, 1/2)$:

- (i) $(3A_y(\delta) - A(\delta))' > 0$.
- (ii) $(3A_x(\delta) - A(\delta))' < 0$.
- (iii) $3A_x(0) - A(0) = 3A_y(0) - A(0) < 0$.
- (iv) $3A_x(\delta) - A(\delta) < 0$.
- (v) $3A_y(1/2) - A(1/2) = 0$.
- (vi) $3(A_y'' - A_x'') + 2\delta(3A_y'' + 3A_x'' - 2A'') + 2(3A_y' + 3A_x' - 2A') > 0$.
- (vii)

$$\frac{3A_y(\delta) - A(\delta)}{3A_x(\delta) - A(\delta)} > \frac{1-2\delta}{1+2\delta}.$$

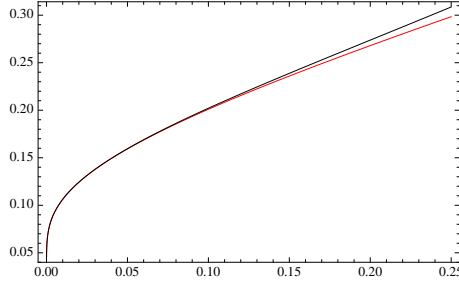
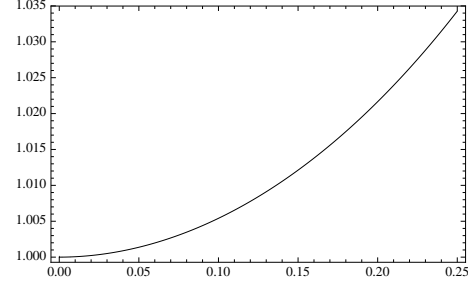
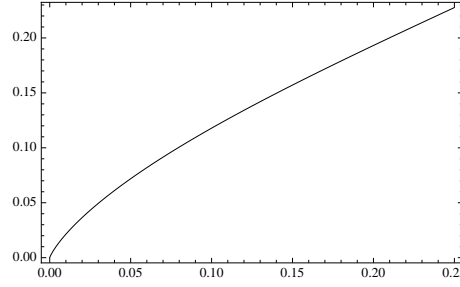
(viii) $3A_x'(0) - A'(0) = -\frac{4\pi}{3\sqrt{3}}$.

(ix) $3A_y'(0) - A'(0) = \frac{4\pi}{3\sqrt{3}}$.

Lemma 4.6. For the positive universal constant η_0 defined in (13.6) and every $\delta \in (0, 1/2)$,

$$\sup_{B_1} |Cp_\delta + \Pi(Z_{p_\delta}, 1/2)| \geq C + \eta_0,$$

for every sufficiently large constant $C > 0$.

FIGURE 3. $1/C_0$ vs. $1/C_y$ FIGURE 4. C_y/C_0 FIGURE 5. $C_x(\delta)/\delta$

Theorem 4.7. *For small $\delta > 0$:*

- (i) $B_y(\delta) = B_y(0) - C_y(\delta)\delta$, where $0 < k_1 \leq C_y(\delta)/|\log \delta| \leq K_2 < +\infty$.
- (ii) $B(\delta) = B(0) - C_0(\delta)\delta$, where $\lim_{\delta \rightarrow 0} C_0(\delta) = +\infty$.
- (iii) $B_x(\delta) = B_x(0) + o(\delta)$ as $\delta \rightarrow 0$.
- (iv) $\lim_{\delta \rightarrow 0} \frac{C_y(\delta)}{C_0(\delta)} = 1$.

5. GROWTH OF THE SOLUTION

Since Z, Z_1, Z_2 have growth $|\mathbf{x}|^2 |\log(|\mathbf{x}|)|$ away from the origin and we expect u (up to some harmonic part) to be close to Z_p , u should share the same growth. We will prove this in the next lemma.

Before we state the lemma let us point out a simple fact that will be used frequently in what follows. By Proposition 3.7 we know that if the origin is a singular point then $u(r\mathbf{x})/r^2 - \Pi(u, r)$ is uniformly bounded by a constant C_0 depending on n and $\|u\|_{L^\infty(B_1)}$. This implies that when $u(r\mathbf{x})/r^2$ is large, say $\max(\sup_{B_1} |\Pi(u, r)|, \sup_{B_1} |u(r\mathbf{x})/r^2|) \geq 2C_0$, then

$$(5.1) \quad \frac{1}{2} \sup_{B_1} |\Pi(u, r)| \leq \sup_{B_1} |u(r\mathbf{x})/r^2| \leq 2 \sup_{B_1} |\Pi(u, r)|.$$

So controlling the size of $\Pi(u, r)$ is equivalent to controlling the size of $u(r\mathbf{x})/r^2$ at singular points.

In the two-dimensional case the following lemma has been proved in [2, Lemma 5.5].

Lemma 5.1. *Let $n = 3$ and let u be a solution to (1.1) in B_1 such that $\sup_{B_1} |u| \leq M$ and $u(0) = |\nabla u(0)| = 0$. Then there exist $\rho_0 > 0$ and $r_0 > 0$ such that if*

$$(5.2) \quad \sup_{B_1} |\Pi(u, r)| \geq \frac{1}{\rho_0}$$

for an $r \leq r_0$ then

$$\sup_{B_1} |\Pi(u, r/2)| > \sup_{B_1} |\Pi(u, r)| + \eta_0/2,$$

where η_0 is the positive constant in Lemma 4.6 (see (13.6) in the Appendix).

Proof. If the Lemma is not true, then there exists a sequence u^j of solutions to (1.1) and $r_j \rightarrow 0$ such that

$$\sup_{B_1} |\Pi(u^j, r_j)| \geq j \text{ and } \sup_{B_1} |\Pi(u^j, r_j/2)| \leq \sup_{B_1} |\Pi(u^j, r_j)| + \eta_0/2.$$

Using Proposition 3.7, and passing if necessary to a subsequence,

$$v^j(\mathbf{x}) := \frac{u^j(r_j \mathbf{x})}{r_j^2} - \Pi(u^j, r_j) \rightarrow v \text{ in } C_{\text{loc}}^{1,\alpha}(\mathbb{R}^3) \cap W_{\text{loc}}^{2,p}(\mathbb{R}^3).$$

We also have $\Pi(u^j, r_j/2) = \Pi(u^j, r_j) + \Pi(v^j, 1/2)$. The limit v satisfies $\Delta v = -\chi_{\{p>0\}}$, where —passing if necessary to another subsequence—

$$p = \lim_{j \rightarrow \infty} p_j \text{ for } p_j := \Pi(u^j, r_j) / \|\Pi(u^j, r_j)\|_{L^\infty(B_1)}.$$

It follows that $\Pi(v, 1/2) = \Pi(Z_p, 1/2)$ where Z_p is the unique solution to

$$\begin{aligned} \Delta Z_p &= -\chi_{\{p>0\}} && \text{in } \mathbb{R}^3, \\ Z_p(0) &= |\nabla Z_p(0)| = \Pi(Z_p, 1) = 0 && \text{and} \\ \lim_{|\mathbf{x}| \rightarrow \infty} \frac{|Z_p(\mathbf{x})|}{|\mathbf{x}|^3} &= 0. \end{aligned}$$

Consequently $\lim_{j \rightarrow \infty} (\Pi(u^j, r_j/2) - \Pi(u^j, r_j)) = \Pi(Z_p, 1/2)$, and

$$(5.3) \quad \sup_{B_1} |\Pi(u^j, r_j/2)| = \sup_{B_1} |\Pi(u^j, r_j) + \Pi(Z_p, 1/2)| + o(1) \text{ as } j \rightarrow \infty.$$

Finally we apply Lemma 4.6 and obtain the statement of the lemma. \square

Remark 5.2. Lemma 5.1 extends to dimension $n > 3$ provided that for some ϵ depending only on n and M

$$\sup_{B_1} \left| \frac{\Pi(u, r)}{\sup_{B_1} |\Pi(u, r)|} - p(Q \cdot) \right| \leq \epsilon$$

for a three dimensional polynomial $p \in \mathbb{P}_2$ and a rotation Q in \mathbb{R}^n .

In the two-dimensional case the following lemma has been proved in [2, Corollary 5.6].

Corollary 5.3. *Let $n = 3$ and let u be a solution to (1.1) in B_1 such that $\sup_{B_1} |u| \leq M$ and $u(0) = |\nabla u(0)| = 0$. Then there exist $\rho_0 > 0$ and $r_0 > 0$ such that if*

$$\sup_{B_1} |\Pi(u, r)| \geq \frac{1}{\rho_0}$$

for an $r \leq r_0$ then

$$\sup_{B_1} |\Pi(u, 2^{-j}r)| \geq \sup_{B_1} |\Pi(u, r)| + j\eta_0/2$$

and

$$\sup_{B_s} |u| \geq \frac{1}{16} \left(\left(\frac{s}{r} \right)^2 \sup_{B_r} |u| + \eta_0 s^2 \log(r/s) \right) \text{ for } 0 < s < r,$$

where η_0 is the positive constant in Lemma 4.6.

Furthermore, there exists a constant $\kappa = \kappa(M, n)$ such that

$$\begin{aligned} \sup_{B_1} |\Pi(u, 2^{-j}r)| &\leq \sup_{B_1} |\Pi(u, r)| + \kappa j, \\ \sup_{B_1} |\Pi(u, s)| &\leq 2 \sup_{B_1} |\Pi(u, r)| + \kappa j \text{ for } s \in (2^{-(j+1)}r, 2^{-j}r], \end{aligned}$$

and

$$\sup_{B_s} |u| \leq 16 \left(\left(\frac{s}{r} \right)^2 \sup_{B_r} |u| + \kappa s^2 \log(r/s) \right).$$

Proof. Lemma 5.1 applies, so that

$$\sup_{B_1} |\Pi(u, 2^{-1}r)| \geq \sup_{B_1} |\Pi(u, r)| + \eta_0/2 \geq \frac{1}{\rho_0}.$$

It follows that Lemma 5.1 applies again with $2^{-1}r$. Thus we may iterate and deduce that

$$\sup_{B_1} |\Pi(u, 2^{-j}r)| \geq \sup_{B_1} |\Pi(u, r)| + j\eta_0/2,$$

which together with (5.1) proves the first part of the Corollary.

We also notice that by Proposition 3.7 we have

$$\sup_{B_1} |\Pi(u, r/2)| \leq \sup_{B_1} |\Pi(u(r\mathbf{x})/r^2, 1/2) - \Pi(u, r)| + \sup_{B_1} |\Pi(u, r)| \leq \kappa + \sup_{B_1} |\Pi(u, r)|.$$

Arguing as above we get

$$\sup_{B_1} |\Pi(u, 2^{-j}r)| \leq \sup_{B_1} |\Pi(u, r)| + j\kappa,$$

which together with (5.1) proves the second part of the Corollary. \square

6. EXISTENCE OF A TRUE THREE-DIMENSIONAL SINGULARITY

Corollary 6.1. *There exists a solution u of (1.1) in $B_1 \subset \mathbb{R}^3$ such that*

$$\lim_{r \rightarrow 0} \frac{u(r\mathbf{x})}{\sup_{B_r} |u|} = \frac{x^2 + y^2}{2} - z^2$$

or

$$\lim_{r \rightarrow 0} \frac{u(r\mathbf{x})}{\sup_{B_r} |u|} = z^2 - \frac{x^2 + y^2}{2}.$$

Proof. The proof is similar to that of [3], so we will only give a sketch. We define the operator $T = T_\epsilon : C^\alpha(B_1^+) \rightarrow C^\alpha(B_1^+)$ by

$$\begin{aligned} \Delta T(u) &= -f_\epsilon(u - u(0)) && \text{in } B_1^+, \\ T(u) &= M\left(\frac{x^2 + y^2}{2} - z^2\right) && \text{on } \partial B_1 \cap \{z > 0\}, \text{ and} \\ \frac{\partial T(u)}{\partial z} &= 0 && \text{on } \{z = 0\} \cap B_1. \end{aligned}$$

Moreover we impose that $T(u)$ has cylindrical symmetry, that is $T(u)(x, y, z) = g(x^2 + y^2, z)$ for some function g . The function $f_\epsilon(t)$ is a smooth approximation of $\chi_{\{t>0\}}$ and M is some large constant.

By Schauder's fixed point theorem there exists an u_ϵ such that $T_\epsilon(u_\epsilon) = u_\epsilon$. We may pass to the limit $\lim_{\epsilon \rightarrow 0} u_\epsilon = \tilde{u}$. Defining $u(\mathbf{x}) = \tilde{u}(\mathbf{x}) - \tilde{u}(0)$ for $z > 0$ and $u(\mathbf{x}) = \tilde{u}(x, y, -z) - \tilde{u}(0)$ for $z < 0$, we see that u solves (1.1). From the boundary condition we infer as in [3] that $\Phi_0^u(r) \leq -M$, which implies that $\sup_{B_1} |\Pi(u, r)|$ is also large. From Corollary 5.3 we conclude therefore that

$$\sup_{B_s} |u| \geq \frac{1}{16} \left(\left(\frac{s}{r} \right)^2 (\sup_{B_r} |u| + \eta_0 s^2 \log(r/s)) \right).$$

But then each limit of

$$\frac{u(r\mathbf{x})}{\sup_{B_r} |u|}$$

as $r \rightarrow 0$ must be a polynomial $p \in \mathbb{P}_2$. Naturally, p will have the same cylindrical symmetry as u . Therefore $p = (x^2 + y^2)/2 - z^2$ or $p = z^2 - (x^2 + y^2)/2$. Last suppose towards a contradiction that there are two subsequences such that one converges to $(x^2 + y^2)/2 - z^2$ and the other to $z^2 - (x^2 + y^2)/2$. By a continuity argument we obtain in this case a third subsequence and a limit that is neither $(x^2 + y^2)/2 - z^2$ nor $z^2 - (x^2 + y^2)/2$, a contradiction. \square

7. ESTIMATING $u - \Pi(u) - Z_{\Pi(u)}$

The following Lemma is a direct consequence of Corollary 4.1 in [13].

Lemma 7.1. *Let p be a second order polynomial in \mathbb{R}^n and $\|p\|_{L^\infty(Q_1)} = 1$. Then*

$$\left(\frac{\mathcal{L}^n(\{|p| \leq \epsilon\})}{|\log(\mathcal{L}^n(\{|p| \leq \epsilon\}))|^{n-1}} \right)^2 \leq C(n)\epsilon \text{ for every } \epsilon \in (0, 1).$$

In particular,

$$\mathcal{L}^n(\{|p| \leq \epsilon\}) \leq C(n, \alpha)\epsilon^\alpha \text{ for every } \epsilon \in (0, 1)$$

and all $\alpha < 1/2$.

The following Lemma is related to the two-dimensional result [2, Lemma 6.1].

Lemma 7.2. *Let u solve (1.1) in $B_1 \subset \mathbb{R}^n$ such that $\sup_{B_1} |u| \leq M$ and $u(0) = |\nabla u(0)| = 0$, and for some $\rho \leq \rho_0$ and $r \leq r_0$ let*

$$\sup_{B_1} |\Pi(u, r)| \geq \frac{1}{\rho}.$$

Furthermore let g_r be the solution of

$$\begin{aligned} \Delta g_r &= \chi_{\{\Pi(u, r) > 0\}} - \chi_{\{u(r \cdot) > 0\}} && \text{in } B_1, \\ g_r &= 0 && \text{on } \partial B_1. \end{aligned}$$

Then for each $\alpha < 1/4$,

- (i) $\|D^2 g_r\|_{L^2(B_1)} \leq C(M, n, \alpha) \sup_{B_1} |\Pi(u, r)|^{-\alpha}.$
- (ii) $\max \left(\sup_{B_1} |\Pi(g_r, 1)|, \sup_{B_1} |\Pi(g_r, 1/2)| \right) \leq C(M, n, \alpha) \sup_{B_1} |\Pi(u, r)|^{-\alpha}.$

Proof. Let $p = \Pi(u, r)$. We know that $\Delta g_r = 1$ when $p > 0$ and $u(r\mathbf{x}) \leq 0$, and that $\Delta g_r = -1$ when $p \leq 0$ and $u(r\mathbf{x}) > 0$; in all other cases it is 0. By Proposition 3.7 we also have that

$$\left| \frac{u(r\mathbf{x})}{r^2} - p \right| \leq C_0.$$

Combining those properties we obtain that $\Delta g_r = 0$ outside the set $\{|p| \leq C_0\}$. From Lemma 7.1 it follows that

$$\|\Delta g_r\|_{L^2(B_1)} \leq (\mathcal{L}^n(\{|p| \leq C_0\}))^{\frac{1}{2}} \leq C(M, n) \sup_{B_1} |p|^{-\alpha} \text{ for each } \alpha < 1/4.$$

Standard L^2 -theory (see for example [21]) thus implies (i).

Rotating and setting $q := \Pi(g_r, t) = \sum_{j=1}^n a_j x_j^2$, where $t = 1$ or $t = 1/2$, we obtain

$$\|D^2 q\|_{L^2(B_1)} \leq C_1 \|D^2 g_r\|_{L^2(B_1)} \leq C_2 \left| \sup_{B_1} |\Pi(u, r)| \right|^{-\alpha}$$

and

$$|a_j| \leq C_3 \left| \sup_{B_1} |\Pi(u, r)| \right|^{-\alpha}$$

for every $1 \leq j \leq n$, proving (ii). \square

The following Lemma is related to the two-dimensional result [2, Lemma 4.3].

Corollary 7.3. *Let u solve (1.1) in $B_1 \subset \mathbb{R}^n$ and assume that $\sup_{B_1} |u| \leq M$, $u(0) = |\nabla u(0)| = 0$, and that for some $\rho \leq \rho_0$ and $r \leq r_0$,*

$$\sup_{B_1} |\Pi(u, r)| \geq \frac{1}{\rho}.$$

Then

$$\sup_{B_1} |\Pi(u, r/2) - \Pi(u, r) - \Pi(Z_{\Pi(u, r)}, 1/2)| \leq C(M, n, \alpha) (\sup_{B_1} |\Pi(u, r)|)^{-\alpha}$$

for each $\alpha < 1/4$.

Proof. For each r write

$$u(r\mathbf{x})/r^2 = \Pi(u, r) + Z_{\Pi(u, r)} + \tilde{g}_r + \tilde{h}_r$$

where $\Delta \tilde{g}_r = \Delta(u(r\mathbf{x})/r^2) - \Delta Z_{\Pi(u, r)}$, $\Delta \tilde{h}_r = 0$ and $\tilde{g}_r(0) = |\nabla \tilde{g}_r(0)| = \tilde{h}_r(0) = |\nabla \tilde{h}_r(0)| = |\Pi(\tilde{g}_r, 1)| = |\Pi(\tilde{h}_r, 1)| = 0$. Next denote by g_r the solution to

$$\begin{aligned} \Delta g_r &= \Delta \tilde{g}_r && \text{in } B_1, \\ g_r &= 0 && \text{on } \partial B_1. \end{aligned}$$

Then $\tilde{g}_r + \tilde{h}_r = g_r + h_r$ for some harmonic function h_r in B_1 . From Lemma 7.2 it follows that

$$\sup_{B_1} |\Pi(g_r, 1/2)| \leq C_1(M, n, \alpha) (\sup_{B_1} |\Pi(u, r)|)^{-\alpha},$$

and from Lemma 3.6 we infer that

$$\begin{aligned} \sup_{B_1} |\Pi(h_r, 1/2)| &= \sup_{B_1} |\Pi(h_r, 1)| = \sup_{B_1} |\Pi(\tilde{g}_r, 1) - \Pi(g_r, 1)| \\ &\leq C_1(M, n, \alpha) (\sup_{B_1} |\Pi(u, r)|)^{-\alpha}. \end{aligned}$$

Thus in B_1

$$\Pi(u, r/2) = \Pi(u, r) + \Pi(Z_{\Pi(u, r)}, 1/2) + \Pi(g_r, 1/2) + \Pi(h_r, 1/2)$$

and

$$|\Pi(u, r/2) - \Pi(u, r) - \Pi(Z_{\Pi(u, r)}, 1/2)| \leq C(M, n, \alpha) (\sup_{B_1} |\Pi(u, r)|)^{-\alpha}. \quad \square$$

8. CLASSIFICATION OF BLOW-UP LIMITS IN \mathbb{R}^3 — AN UNEXPECTED SYMMETRIZATION EFFECT

In this section we will show that if $\lim_{j \rightarrow \infty} \frac{u(r_j \mathbf{x})}{\sup_{B_{r_j}} |u|} = p$, where p is a harmonic polynomial, then $p = 2xz$, $p = (x^2 + y^2)/2 - z^2$ or $p = z^2 - (x^2 + y^2)/2$ up to a rotation.

Theorem 8.1. *Let $n = 3$, let $\Delta u = -\chi_{\{u > 0\}}$ in B_1 and assume that $u(0) = |\nabla u(0)| = 0$ and that the monotonicity energy satisfies $\lim_{r \rightarrow 0} \Phi_{\mathbf{x}^0}^u(r) = -\infty$. Then each limit of*

$$\frac{u(r\mathbf{x})}{r^2} - \Pi(u(\mathbf{x}), r),$$

as $r \rightarrow 0$, is contained in

$$\{Z(Q\mathbf{x}) : Q \in \mathcal{Q}\} \cup \{Z_1(Q\mathbf{x}) : Q \in \mathcal{Q}\} \cup \{Z_2(Q\mathbf{x}) : Q \in \mathcal{Q}\};$$

here \mathcal{Q} is the set of all rotations of \mathbb{R}^3 .

Proof. Suppose towards a contradiction that the statement is not true. By Proposition 3.2 (i) there exists a solution u and a sequence $r_j \rightarrow 0$ such that after rotation

$$(8.1) \quad \lim_{j \rightarrow \infty} \left| \frac{u(r_j \mathbf{x})}{\sup_{B_{r_j}} |u|} - p_{\delta_0} \right| = 0$$

for some $\delta_0 \in (0, 1/2)$ and $p_{\delta_0} = (1/2 + \delta_0)x^2 + (1/2 - \delta_0)y^2 - z^2$ and $\delta_0 \in (0, 1/2)$ or $p_{\delta_0} = z^2 - (1/2 + \delta_0)x^2 - (1/2 - \delta_0)y^2$. We may assume that $p_{\delta_0} = (1/2 + \delta_0)x^2 + (1/2 - \delta_0)y^2 - z^2$. Furthermore, from Proposition 3.2 (i) and Proposition 3.7, $\lim_{r \rightarrow 0} \sup_{B_1} |u(r\mathbf{x})/r^2| = \infty$.

We are going to prove a decay estimate for $\delta_r^A = \delta^A(u(r \cdot))$ in r which will lead to a contradiction to (8.1).

By Theorem 4.5 (vii),

$$\frac{3A_y(\delta) - A(\delta)}{3A_x(\delta) - A(\delta)} > \frac{1 - 2\delta}{1 + 2\delta} \text{ for } \delta \in (0, 1/2).$$

Thus

$$(8.2) \quad \kappa(\delta) := (1 + 2\delta) \frac{3A_y(\delta) - A(\delta)}{3A_x(\delta) - A(\delta)} - (1 - 2\delta) \geq \omega(\delta) > 0,$$

where ω is a continuous function on $[0, 1/2]$.

By Corollary 7.3, using Corollary 5.3 to estimate

$$(\sup_{B_1} |\Pi(u, r)|)^{-\alpha} = O(|\log(r)|^{-\alpha}),$$

we obtain for every $\alpha < 1/4$ that in B_1 , up to a rotation depending on r ,

$$(8.3) \quad |\Pi(u, r/2) - \tau_r p_{\delta_r^A} - \Pi(Z_{\delta_r^A}, 1/2)| \leq O(|\log(r)|^{-\alpha});$$

from here on, Z_δ is the unique solution to

$$\begin{aligned} \Delta Z_\delta &= -\chi_{\{p_\delta > 0\}} \\ Z_\delta(0) &= |\nabla Z_\delta(0)| = \Pi(Z_\delta, 1) = \lim_{|\mathbf{x}| \rightarrow \infty} Z_\delta(\mathbf{x})/|\mathbf{x}|^3 = 0. \end{aligned} \quad \text{in } \mathbb{R}^3$$

In particular, for the \mathbf{K}_0 defined in (4.1),

$$\begin{aligned} \Pi(Z_\delta, 1/2) &= -\mathbf{K}_0[(3A_x(\delta) - A(\delta))x^2 + (3A_y(\delta) - A(\delta))y^2 + (3A_z(\delta) - A(\delta))z^2] \\ &= -\mathbf{K}_0 K_1[(1 + 2\delta)x^2 + K_2(1 + 2\delta)y^2 + K_3(1 + 2\delta)z^2], \end{aligned}$$

where — using the fact that $\Pi(Z_\delta, 1/2)$ is harmonic —

$$\begin{aligned} K_1 &= \frac{3A_x(\delta) - A(\delta)}{(1 + 2\delta_0)}, \\ K_2 &= \frac{3A_y(\delta) - A(\delta)}{3A_x(\delta) - A(\delta)} = \frac{1 - 2\delta + \kappa(\delta)}{1 + 2\delta} \end{aligned}$$

and

$$K_3 = \frac{3A_z(\delta) - A(\delta)}{3A_x(\delta) - A(\delta)} = -(1 + K_2) = -\frac{2 + \kappa(\delta)}{1 + 2\delta}.$$

It follows that

$$\begin{aligned} \Pi(Z_\delta, 1/2) &= -\mathbf{K}_0 K_1[(1 + 2\delta)x^2 + (1 - 2\delta)y^2 + \kappa(\delta)y^2 - 2z^2 - \kappa(\delta)z^2] \\ &= c_\delta(2p_\delta + \kappa(\delta)(y^2 - z^2)) \end{aligned}$$

for $c_\delta = -\mathbf{K}_0(3A_x(\delta) - A(\delta))/(1 + 2\delta) \geq \bar{c} > 0$, $\delta \in [0, 1/2]$ (see Theorem 4.5).

Invoking (8.3), this implies that in B_1 , up to a rotation depending on r ,

$$(8.4) \quad |\Pi(u, r/2) - \tau_r p_{\delta_r^A} - c_{\delta_r^A}(2p_{\delta_r^A} + \kappa(\delta_r^A)(y^2 - z^2))| = O(|\log(r)|^{-\alpha}).$$

The fact that $c_\delta \geq \bar{c} > 0$ as well as the estimate $\kappa \geq \omega > 0$ consequently prove together with Corollary 5.3 that, rotating slightly,

$$(8.5) \quad \delta_{r/2}^A \leq \delta_r^A - c_1 \frac{\omega(\delta_r^A)}{|\log r|} + C_2 |\log(r)|^{-1-\alpha}.$$

Note that estimate (8.5) is independent of rotations. As long as $\omega(\delta_{2^{-k}r_0}^A) \geq |\log(2^{-k}r_0)|^{-\alpha/2}$, induction of estimate (8.5) in k yields a logarithmic decay of δ_r^A in r . It follows that $\delta_r^A \rightarrow 0$ as $r \rightarrow 0$, contradicting the assumption $\lim_{j \rightarrow \infty} \delta_{r_j}^A = \delta_0 > 0$. \square

9. UNIQUE TANGENT CONES AT TRUE THREE DIMENSIONAL SINGULARITIES

From the previous section we may infer by a continuity argument that in three dimensions, assuming $u(0) = |\nabla u| = 0$ as well as $\lim_{r \rightarrow 0} \Phi_{\mathbf{x}^0}^u(r) = -\infty$, then one of the following three statements holds:

$$\begin{aligned} (i) \quad & \lim_{r \rightarrow 0} \left(\frac{u(rQ(r)\mathbf{x})}{r^2} - \Pi(u, r)(Q(r)\mathbf{x}) \right) = Z(\mathbf{x}), \\ (ii) \quad & \lim_{r \rightarrow 0} \left(\frac{u(rQ(r)\mathbf{x})}{r^2} - \Pi(u, r)(Q(r)\mathbf{x}) \right) = Z_1(\mathbf{x}), \\ (iii) \quad & \lim_{r \rightarrow 0} \left(\frac{u(rQ(r)\mathbf{x})}{r^2} - \Pi(u, r)(Q(r)\mathbf{x}) \right) = Z_2(\mathbf{x}) \end{aligned}$$

for some $Q(r) \in \mathcal{Q}$. However at this point we do not yet know whether the rotation $Q(r)$ converges as $r \rightarrow 0$.

In this section we will show that in the case (ii) and (iii), $\frac{u(r\mathbf{x})}{r^2} - \Pi(u, r)(\mathbf{x})$ converges as $r \rightarrow 0$. In Section 11 we will show a similar result in the case (i).

Theorem 9.1. *Let $n = 3$ and let u solve $\Delta u = -\chi_{\{u>0\}}$ in B_1 such that $u(0) = |\nabla u(0)| = 0$ and $M := \sup_{B_1} |u| < +\infty$. There exist constants $r(M) > 0, c(M) > 0$ and $K(M) < +\infty$ such that if*

$$s \in (0, r(M)), \quad \Pi(u, s) \geq K(M) \text{ and } \delta^A(u(s\mathbf{x})) \leq c(M),$$

then there is a rotation Q such that either

$$(9.1) \quad \frac{u(r\mathbf{x})}{r^2} - \Pi(u, r)(\mathbf{x}) \rightarrow Z_1(Q\mathbf{x}) \text{ as } r \rightarrow 0$$

or

$$(9.2) \quad \frac{u(r\mathbf{x})}{r^2} - \Pi(u, r)(\mathbf{x}) \rightarrow Z_2(Q\mathbf{x}) \text{ as } r \rightarrow 0.$$

Moreover, there exist $\beta > 0$ and $C(M, \beta) < +\infty$ such that

$$\sup_{B_1} \left| \frac{\Pi(u, r)}{\sup_{B_1} |\Pi(u, r)|} - \frac{p}{\sup_{B_1} |p|} \right| \leq C(M, \beta) \left(K(M) + \left| \log \left(\frac{r}{s} \right) \right| \right)^{-\beta}$$

for all $r \in (0, s)$; here $p(\mathbf{x}) = (x^2 + y^2)/2 - z^2$ in the case (9.1) and $p(\mathbf{x}) = -(x^2 + y^2)/2 - z^2$ in the case (9.2).

Proof. Observe that the assumptions imply by Corollary 5.3 as in the proof of Corollary 6.1 that $\tau_r \geq K(M)$ for $r < s$ and that $\tau_r \rightarrow +\infty$ as $r \rightarrow 0$. Moreover we see from Theorem 8.1 that $\frac{u(s\mathbf{x})}{s^2} - \Pi(u, s)(\mathbf{x})$ is after rotation close to $Z_1(Q\mathbf{x})$ or $Z_2(Q\mathbf{x})$. We may assume that it is close to $Z_1(\mathbf{x})$.

We will follow the strategy explained in the proof of Theorem 8.1, and use the notation of that proof. Remember that by (8.4) and Corollary 5.3, up to a rotation depending on r ,

$$(9.3) \quad |\Pi(u, r/2) - \tau_r p_{\delta_r^A} - c_{\delta_r^A} (2p_{\delta_r^A} + \kappa(\delta_r^A)(y^2 - z^2))| = O(\tau_r^{-\alpha}),$$

where $c_\delta \geq \bar{c} > 0$ and

$$(9.4) \quad \kappa(\delta) = (1 + 2\delta) \frac{3A_y(\delta) - A(\delta)}{3A_x(\delta) - A(\delta)} - (1 - 2\delta) \geq \omega(\delta) > 0.$$

In Theorem 8.1 we worked to exclude the case that $\delta^A(u(r\cdot)) \in [\beta, 1/2 - \beta]$ for positive β and small r , and in that δ -regime, ω has been bounded from below by a positive constant. In the present proof, however, we are interested in the regime $\delta \rightarrow 0$, where ω degenerates. In order to deal with this difficulty, we will make a Taylor expansion of $3A_x(\delta) - A(\delta)$ and $3A_y(\delta) - A(\delta)$ at the point $\delta = 0$: From Theorem 4.5 we infer that

$$(9.5) \quad 3A_x(\delta) - A(\delta) = 3A_x(0) - A(0) - \frac{4\pi}{3\sqrt{3}}\delta + O(\delta^2)$$

and

$$(9.6) \quad 3A_y(\delta) - A(\delta) = 3A_y(0) - A(0) + \frac{4\pi}{3\sqrt{3}}\delta + O(\delta^2).$$

Plugging this information into (9.4), we obtain that

$$(9.7) \quad 4\delta \leq \kappa(\delta) \leq \left(5 + \frac{16\pi}{2\sqrt{3}}\right) \delta.$$

Dividing (9.3) by τ_r , rotating slightly and recalling that $p_\delta = (1/2 + \delta)x^2 + (1/2 - \delta)y^2 - z^2$ and using that $c_\delta \geq \bar{c} > 0$ we infer that

$$(9.8) \quad \delta_{r/2}^A \leq \delta_r^A - c_1 \frac{\delta_r^A}{\tau_r} + C_2 \tau_r^{-1-\alpha},$$

where $c_1 > 0$ is a universal constant and $C_2 < +\infty$ depends only on M and α . Note that estimate (9.8) is independent of rotations. In the following three Claims we will describe how (9.8) leads to a decay estimate for δ^A .

Claim 1: *There is a universal constant $\beta > 0$ and $C_3 = C_3(M) < +\infty$ such that if $\tau_r \geq C(M)$ and $\delta_r^A \leq \tau_r^{-\beta}$, then*

$$\delta_{r/2}^A \leq \tau_{r/2}^{-\beta}.$$

Proof of Claim 1: First, (9.8) as well as the assumption in the Claim imply that

$$\delta_{r/2}^A \leq (1 - (c_1 - C_2 \tau_r^{-\alpha-\beta}) \tau_r^{-1}) \tau_r^{-\beta}.$$

On the other hand, $0 \leq \tau_{r/2} - \tau_r \leq \kappa$ (see Corollary 5.3), so that

$$(9.9) \quad \left(\frac{\tau_{r/2}}{\tau_r}\right)^{-\beta} \geq 1 - \beta \kappa \tau_r^{-1}.$$

It follows that, provided that β has been chosen small enough (depending only on the universal constants κ and c_1) and τ_r is large enough (depending on κ, c_1 and M), then

$$\delta_{r/2}^A \leq \tau_{r/2}^{-\beta},$$

proving the claim.

Next we consider the case when $\delta_r^A \geq \tau_r^{-\beta}$.

Claim 2: *There is a constant $\beta = \beta(\alpha) \in (0, \alpha)$ such that if*

$$\delta_r^A \leq c_4(M), \quad \tau_r \geq C_5 \text{ and } \delta_r^A \geq \tau_r^{-\beta},$$

then

$$(9.10) \quad \delta_{r/2}^A \leq \left(1 - \frac{c_1}{2\tau_r}\right) \delta_r^A.$$

Moreover, if $\delta_{2^{-k}r}^A \geq \tau_{2^{-k}r}^{-\beta}$ for each $k \leq k_0$ then

$$\delta_{2^{-k_0}r}^A \leq \frac{\tau_r^{2\beta} \delta_r^A}{\tau_{2^{-k_0}r}^\beta} \frac{1}{\tau_{2^{-k_0}r}^\beta}.$$

Proof of Claim 2: Equation (9.10) is a direct consequence of equation (9.8). The last part of the Claim follows from an induction of the first part (noting that the assumption $\delta_r^A \leq c_4(M)$ is satisfied inductively): If $\delta_{2^{-k}r}^A \geq \tau_{2^{-k}r}^{-\beta}$ for each $k \leq k_0$ then

$$\delta_{2^{-k_0}r}^A \leq \delta_r^A \prod_{k=0}^{k_0-1} \left(1 - \frac{c_1}{2\tau_{2^{-k}r}}\right).$$

That product can be estimated for $\tau_r \geq C_5$, calculating

$$\begin{aligned} \log \left(\prod_{k=0}^{k_0-1} \left(1 - \frac{c_1}{2\tau_{2^{-k}r}} \right) \right) &= \sum_{k=0}^{k_0-1} \log \left(1 - \frac{c_1}{2\tau_{2^{-k}r}} \right) \leq - \sum_{k=0}^{k_0-1} \frac{c_1}{4\tau_{2^{-k}r}} \\ &\leq - \sum_{k=0}^{k_0-1} \frac{c_1}{4(\tau_r + \kappa k)} \leq -\frac{c_1}{4} \log \left(\frac{k_0\kappa + \tau_r}{\tau_r} \right), \end{aligned}$$

where we have used Corollary 5.3. Thus

$$\prod_{k=0}^{k_0-1} \left(1 - \frac{c_1}{2\tau_{2^{-k}r}} \right) \leq \left(\frac{\tau_r}{k_0\kappa + \tau_r} \right)^{c_1/4}.$$

Choosing β even smaller such that $2\beta \leq c_1/4$ and using once more Corollary 5.3, we obtain

$$\begin{aligned} \delta_{2^{-k_0}r}^A &\leq \left(\frac{\tau_r}{k_0\kappa + \tau_r} \right)^{2\beta} \delta_r^A \\ &\leq \left(\frac{1}{k_0\kappa + \tau_r} \right)^\beta \frac{\tau_r^{2\beta} \delta_r^A}{(k_0\kappa + \tau_r)^\beta} \leq \frac{\tau_r^{2\beta} \delta_r^A}{\tau_{2^{-k_0}r}^\beta \tau_{2^{-k_0}r}^\beta} \frac{1}{\tau_{2^{-k_0}r}^\beta}, \end{aligned}$$

proving the Claim.

Claim 3: *There is a constant $\beta = \beta(\alpha) \in (0, \alpha)$ such that if $\tau_{2^{-k_0}r} \geq C(M)$ and $\delta_r^A \leq c_4(M)$, then for $k \geq k_0$,*

$$\begin{aligned} \delta_{2^{-k}r}^A &\leq \tau_{2^{-k}r}^{-\beta} \text{ for } k \geq k_1 \text{ and} \\ \delta_{2^{-k}r}^A &\leq \frac{\tau_r^{2\beta} \delta_r^A}{\tau_{2^{-k}r}^{2\beta}} \text{ for } k < k_1 \\ \text{for some } k_1 &\leq \frac{4}{\eta_0} \tau_r^2 (\delta_r^A)^{\frac{1}{\beta}}. \end{aligned}$$

Proof of Claim 3: We apply Claim 2 up to the first k_1 such that $\delta_{2^{-k_1}r}^A \leq \tau_{2^{-k_1}r}^{-\beta}$, and we apply Claim 1 for $k \geq k_1$. From Claim 2 and Corollary 5.3 we infer that

$$k_1 \leq \frac{4}{\eta_0} \tau_r^2 (\delta_r^A)^{\frac{1}{\beta}}.$$

Observing that the assumptions for $\tau_{2^{-k}r}$ are satisfied for $k \geq k_0$ by Corollary 5.3 finishes the proof of Claim 3.

In the last part of our proof we will use the decay estimate in Claim 3 in order to estimate how much $\Pi(u(r \cdot))$ moves when varying r . Let $r_k := 2^{-k}s$, $\tau_k := \tau_{2^{-k}s}$ and $\delta_k^A := \delta_{2^{-k}s}^A$. First, we infer from (9.3) that up to a rotation depending on k ,

$$\begin{aligned} (9.11) \quad \sup_{B_1} \left| \frac{\Pi(u, r_k)}{\tau_k} - \frac{\Pi(u, r_{k+1})}{\tau_{k+1}} \right| \\ \leq \sup_{B_1} \left| \frac{\tau_k p_{\delta_k^A}}{\tau_k} - \frac{\tau_k p_{\delta_k^A} + c_{\delta_k^A} (2p_{\delta_k^A} + \kappa(\delta_k^A)(y^2 - z^2))}{\tau_{k+1}} \right| + C_1(M, \alpha) \tau_k^{-(1+\alpha)}, \end{aligned}$$

where $0 \leq c_{\delta_k^A} = -\mathbf{K}_0(3A_x(\delta_k^A) - A(\delta_k^A))/(1 + 2\delta_k^A) \leq C_6$, $4\delta_k^A \leq \kappa(\delta_k^A) \leq C_7\delta_k^A$ and C_6, C_7 are universal constants. Another fact we infer from (9.3) is that

$$(9.12) \quad \tau_{k+1} = \tau_k + 2c_{\delta_k^A} + O(\delta_k^A).$$

Plugging (9.12) into (9.11) yields

$$\begin{aligned} & \sup_{B_1} \left| \frac{\Pi(u, r_k)}{\tau_k} - \frac{\Pi(u, r_{k+1})}{\tau_{k+1}} \right| \\ & \leq \sup_{B_1} \left| \frac{-4c_{\delta_k^A}^2 + \tau_k O(\delta_k^A)}{\tau_k(\tau_k + 2c_{\delta_k^A} + O(\delta_k^A))} \right| + C_1(M, \alpha) \tau_k^{-(1+\alpha)} \leq C_8(M, \alpha) \tau_k^{-(1+\alpha)} + C_9 \frac{\delta_k^A}{\tau_k}, \end{aligned}$$

where C_9 is a universal constant. Iterating this estimate we obtain

$$(9.13) \quad \sup_{B_1} \left| \frac{\Pi(u, r_k)}{\tau_k} - \frac{\Pi(u, r_{k+m})}{\tau_{k+m}} \right| \leq \sum_{i=k}^{k+m} \frac{C_8(M, \alpha)}{\tau_i^{1+\alpha}} + \sum_{i=k}^{k+m} C_9 \frac{\delta_i^A}{\tau_i}.$$

From Claim 3 (applied twice) and Corollary 5.3 we conclude that, choosing $r(M)$ small enough such that $\tau_{r(M)} \geq C(M)$, setting $p = x^2 + y^2 - 2z^2$ and letting $m_j \rightarrow \infty$,

$$\begin{aligned} & \sup_{B_1} \left| \frac{\Pi(u, r_k)}{\tau_k} - p \right| \leq C_{10}(M, \beta) \sum_{i=k}^{\infty} \tau_i^{-1-\beta} + C_9 \tau_s^{2\beta} \delta_s^A \sum_{i=k}^k \tau_i^{-1-2\beta} \\ & \leq C_{11}(M, \beta) \sum_{i=k}^{\infty} (M + i\eta_0/2)^{-1-\beta} + C_{12} \tau_s^{2\beta} \delta_s^A \sum_{i=k}^{\infty} (M + i\eta_0/2)^{-1-2\beta} \\ & \leq C_{13}(M, \beta) \tau_s^{-\beta} \text{ for all } k \geq k(M). \end{aligned}$$

Using once more Corollary 5.3 we obtain the estimate of the Theorem as well as

$$\lim_{r \rightarrow 0} \left(\frac{u(\mathbf{x}^0 + r\mathbf{x})}{r^2} - \Pi(u, r, \mathbf{x}^0)(\mathbf{x}) \right) = Z_1(\mathbf{x}) \text{ for } \mathbf{x}^0 = 0.$$

□

Corollary 9.2. *Let $n = 3$, let u solve (1.1) in B_1 and suppose that*

$$(9.14) \quad \lim_{r \rightarrow 0} \frac{u(r\mathbf{x})}{\sup_{B_r} |u|} = (x^2 + y^2)/2 - z^2 \text{ or } \lim_{r \rightarrow 0} \frac{u(r\mathbf{x})}{\sup_{B_r} |u|} = z^2 - (x^2 + y^2)/2.$$

Then there exists an $r_0 = r_0(u)$ and $f, g \in C^{0,1}(B'_{r_0})$ such that

$$B_{r_0} \cap \{u = 0\} = \{(x, y, f(x, y)) : (x, y) \in B'_{r_0}\} \cup \{(x, y, g(x, y)) : (x, y) \in B'_{r_0}\} \cap B_{r_0}.$$

Moreover $f(x, y) - \sqrt{x^2 + y^2}/\sqrt{2} \in C^1(B'_{r_0})$ and $g(x, y) + \sqrt{x^2 + y^2}/\sqrt{2} \in C^1(B'_{r_0})$. The Lipschitz- and C^1 -norms corresponding to the above statements are uniformly bounded for solutions v sufficiently close to the fixed solution u in $L^\infty(B_1)$, provided that each v satisfies

$$\lim_{r \rightarrow 0} \frac{v(\xi^v + rQ^v \mathbf{x})}{\sup_{B_r(\xi^v)} |v|} = \lim_{r \rightarrow 0} \frac{u(r\mathbf{x})}{\sup_{B_r} |u|}$$

for some rotation Q^v at a singular point ξ^v sufficiently close to the origin.

Proof. We will show that $\{u = 0\} \cap B_{r_0}^+ = \{(x, y, f(x, y)) : (x, y) \in B'_{r_0}\} \cap B_{r_0}$ for some $f \in C^{0,1}(B'_{r_0})$ and $f - \sqrt{x^2 + y^2}/\sqrt{2} \in C^1$. By symmetry a similar statement holds in $B_{r_0}^-$. We will also assume, for the sake of definiteness, that

$$v_r(\mathbf{x}) = \frac{u(r\mathbf{x})}{\sup_{B_r} |u|} \rightarrow (x^2 + y^2)/2 - z^2 \text{ in } C^{1,\alpha}(\overline{B_1}).$$

By the $C^{1,\alpha}$ -convergence,

$$\sup_{B_1} \left| \frac{\partial v_r}{\partial z} + 2z \right| \leq \omega(r)$$

and

$$\sup_{B_1} \left| v_r - \left(\frac{x^2 + y^2}{2} - z^2 \right) \right| \leq \omega(r)$$

for some modulus of continuity $\omega(r) \rightarrow 0$ as $r \rightarrow 0$. It follows that $\{v_r = 0\} \cap (\bar{B}_1 \setminus B_{1/2}) \subset \{(x, y, z) \in \bar{B}_1 \setminus B_{1/2} : \text{dist}(\cdot, \{x^2 + y^2 = 2z^2\}) \leq \sigma(r)\}$ for some modulus of continuity σ . Therefore

$$\frac{\partial v_r}{\partial z} \leq -\frac{1}{4} \text{ on } \bar{B}_1^+ \setminus B_{1/2}.$$

From the implicit function theorem and $C^{1,\alpha}$ -regularity we infer that $\{v_r = 0\} \cap (\bar{B}_1^+ \setminus B_{1/2})$ is a $C^{1,\alpha}$ -graph with bounded $C^{1,\alpha}$ -norm (independent of r). It follows that $\{u = 0\}$ is the graph of a Lipschitz function f in $B_{r_0}^+$ and we only need to show that $f(x, y) - \sqrt{x^2 + y^2}/\sqrt{2} \in C^1(B_{r_0})$.

We know that $f \in C^{1,\alpha}(\bar{B}_{r_0}' \setminus B_s')$ for every $s > 0$ and that $f(rx, ry)/r$ is bounded in $C^{1,\alpha}(\bar{B}_1' \setminus B_{1/2}')$. Thus it is sufficient to show that

$$\lim_{(x,y) \rightarrow (0,0)} |\nabla(\sqrt{2}f(x, y) - \sqrt{x^2 + y^2})| = 0.$$

Let us consider any sequence $(x_j, y_j) \rightarrow 0$ and denote $\sqrt{x_j^2 + y_j^2} = r_j$. Then $f(r_j x, r_j y)/r_j$ will converge to $\sqrt{x^2 + y^2}/\sqrt{2}$ in $C^{1,\alpha}(\bar{B}_1' \setminus B_{1/2}')$, implying that

$$\left(\nabla f(x, y) - \nabla \frac{\sqrt{x^2 + y^2}}{\sqrt{2}} \right) \Big|_{(x,y)=(x_j,y_j)} \rightarrow 0.$$

As the sequence (x_j, y_j) is arbitrary, it follows that $f - \sqrt{x^2 + y^2}/\sqrt{2} \in C^1$. The uniformity follows from the uniformity in Theorem 9.1. \square

10. STABLE CONES

Theorem 10.1. *In \mathbb{R}^3 there exists a solution of (1.1) in B_1 such that*

$$(10.1) \quad 0 \leq \frac{1}{2} \int_{B_1} |\nabla w|^2 \leq \int_{B_1} |\nabla w|^2 - \int_{B_1 \cap \{u=0\}} \frac{w^2}{|\nabla u|} d\mathcal{H}^2$$

for each $w \in W_0^{1,2}(B_1)$. Moreover $u \notin C^{1,1}(B_{1/2})$.

Notice that the right-hand side in (10.1) is the second variation of the energy $\int_{B_1} (|\nabla u|^2/2 - \max(u, 0))$ of equation (1.1).

Proof. By Corollary 6.1 there exists a solution v of $\Delta v = -\chi_{\{v>0\}}$ in B_1 such that the blow-up limit at the origin is Z_1 .

Let

$$u(\mathbf{x}) := \frac{v(s\mathbf{x})}{s^2}$$

for some small but fixed s .

For some large M to be determined later and sufficiently small s , Theorem 9.1 together with Corollary 5.3 yields that

$$(10.2) \quad |\nabla u(\mathbf{x})| \geq (M + \log(\frac{1}{|\mathbf{x}|}))|\mathbf{x}|$$

on $\Gamma = B_1 \cap \{u = 0\}$.

Choosing s if necessary even smaller, Corollary 9.2 implies that Γ consists of two Lipschitz graphs in $\overline{B_1}$.

Note that since the origin has zero capacity we may by a limiting argument deduce that the second variation is well defined for all $w \in W_0^{1,2}(B_1)$.

If $w \in W^{1,2}$ then $w|_\Gamma \in W^{1/2,2}(\Gamma)$ by the trace theorem, which is valid for our Lipschitz free boundary. Also from the trace theorem, combined with Poincaré's inequality, we infer that for each $w \in W_0^{1,2}(B_1)$

$$\|w\|_{W^{1/2,2}(\Gamma)} \leq C_1 \|w\|_{W^{1,2}(B_1)} \leq C_2 \|\nabla w\|_{L^2(B_1)}.$$

Using the Sobolev embedding, we obtain for $w \in W_0^{1,2}(B_1)$

$$(10.3) \quad \|w\|_{L^4(\Gamma)} \leq C_3 \|w\|_{W^{1/2,2}(\Gamma)} \leq C_4 \|\nabla w\|_{L^2(B_1)}.$$

Thus, using (10.2) and (10.3),

$$\begin{aligned} \int_\Gamma \frac{w^2}{|\nabla u|} d\mathcal{H}^2 &\leq \left(\int_\Gamma \frac{1}{|\nabla u|^2} d\mathcal{H}^2 \right)^{1/2} \left(\int_\Gamma w^4 d\mathcal{H}^2 \right)^{1/2} \\ &\leq C_5 \left(\int_\Gamma \frac{1}{(M + (\log(|\mathbf{x}|^{-1}))^2)|\mathbf{x}|^2} d\mathcal{H}^2 \right)^{1/2} \int_{B_1} |\nabla w|^2. \end{aligned}$$

On the other hand, $|\Gamma \cap \partial B_r| \leq C_6 r$ by Corollary 9.2, so that

$$\left(\int_\Gamma \frac{1}{(M + (\log(|\mathbf{x}|^{-1}))^2)|\mathbf{x}|^2} d\mathcal{H}^2 \right)^{1/2} \leq C_7 \left(\int_0^1 \frac{r}{(M + (\log |r|)^2)r^2} \right)^{1/2} \leq \frac{C_8}{\sqrt{M}}.$$

Choosing $M \geq 4C_8^2 C_5^2$ we arrive at

$$\int_\Gamma \frac{w^2}{|\nabla u|} d\mathcal{H}^2 \leq \frac{1}{2} \int_{B_1} |\nabla w|^2.$$

□

11. UNIQUE TANGENT CONES AT UNSTABLE CODIMENSION 2 SINGULARITIES

In Theorem 9.1 we showed that if

$$\lim_{j \rightarrow \infty} \frac{u(r_j \mathbf{x})}{\sup_{B_{r_j}} |u|} = (x^2 + y^2)/2 - z^2,$$

then the blow-up limit is unique and we obtain a quantitative convergence estimate. In this section we will show the corresponding result in the case that

$$\lim_{j \rightarrow \infty} \frac{u(r_j \mathbf{x})}{\sup_{B_{r_j}} |u|} = x^2 - z^2.$$

This case corresponds to $\delta = 1/2$ in the notation of the previous sections. To make Taylor expansions of $3A_x(\delta) - A(\delta)$ etc. around the point $\delta = 1/2$ would be rather clumsy. To get around that we will change the parametrization to $p_\delta = (1 - \delta)x^2 + \delta y^2 - z^2$ and use the B_x, B_y, B_z, B defined in Section 4.

Theorem 11.1. *Let $n = 3$, let u solve $\Delta u = -\chi_{\{u>0\}}$ in B_1 and suppose that $M := \sup_{B_1} |u| < +\infty$, $\mathbf{x}^0 \in B_{1/2}$, $u(\mathbf{x}^0) = |\nabla u(\mathbf{x}^0)| = 0$ and that there exists a sequence $r_j \rightarrow 0$ such that*

$$\lim_{j \rightarrow \infty} \left(\frac{u(\mathbf{x}^0 + r_j \mathbf{x})}{r_j^2} - \Pi(u, r_j, \mathbf{x}^0)(\mathbf{x}) \right) = Z(Q_{\mathbf{x}^0} \mathbf{x}),$$

where $Q_{\mathbf{x}^0}$ is a rotation depending on the point \mathbf{x}^0 . Then the limit

$$\lim_{r \rightarrow 0} \left(\frac{u(\mathbf{x}^0 + r \mathbf{x})}{r^2} - \Pi(u, r, \mathbf{x}^0)(\mathbf{x}) \right) = Z(Q_{\mathbf{x}^0} \mathbf{x})$$

exists (and is thus unique).

Moreover, for each $\gamma \in (0, 1/4)$ there exist constants $r(M, \gamma) > 0$, $c(M) > 0$, $K(M) < +\infty$ and $C(M, \gamma) < +\infty$ such that

$$s \in (0, r(M, \gamma)), \quad \Pi(u, s) \geq K(M) \text{ and } \delta^B(u(s\mathbf{x})) \leq c(M)$$

imply that

$$\left| \frac{\Pi(u, r, \mathbf{x}^0)(\mathbf{x})}{\sup_{B_1} |\Pi(u, r, \mathbf{x}^0)|} - 2(Q_{\mathbf{x}^0} \mathbf{x})_1 (Q_{\mathbf{x}^0} \mathbf{x})_3 \right| \leq C(M, \gamma) \left(K(M) + \left| \log \left(\frac{r}{s} \right) \right| \right)^{-\gamma}$$

for all $r \in (0, s)$.

Proof. We may assume that $\mathbf{x}^0 = 0$ and that the rotation $Q_{\mathbf{x}^0}$ is such that

$$2(Q_{\mathbf{x}^0} \mathbf{x})_1 (Q_{\mathbf{x}^0} \mathbf{x})_3 = \mathbf{x}_1^2 - \mathbf{x}_3^2.$$

By Corollary 7.3, up to a rotation depending on r ,

$$(11.1) \quad \sup_{B_1} |\Pi(u, r/2) - \tau_r p_{\delta_r^B} - \Pi(Z_{p_{\delta_r^B}}, 1/2)| \leq C(M, n, \gamma) \tau_r^{-\gamma}$$

for $\gamma < 1/4$ and $r < r(M)$. Following the strategy in the proof of Theorem 9.1, we are going to use (11.1) together with an analysis of $\Pi(Z_{p_{\delta_r^B}}, 1/2)$ to derive a decay estimate for $\delta^B(u(r\mathbf{x})) \geq 0$ (cf. Definition 4.2) in r . That decay in turn will make it possible to estimate how much $\Pi(u(r\cdot))$ moves when decreasing r . Note however that as the singularity examined in the present section is by [15] *unstable*, we cannot expect to obtain the decay by a simple iteration as in the proof of Theorem 9.1. The “pinning effect” of the convergence assumption

$$\lim_{j \rightarrow \infty} \left(\frac{u(\mathbf{x}^0 + r_j \mathbf{x})}{r_j^2} - \Pi(u, r_j, \mathbf{x}^0)(\mathbf{x}) \right) = Z(Q_{\mathbf{x}^0} \mathbf{x}),$$

has to enter the proof.

Claim: For $r < s$,

$$\delta_r^B \leq \frac{2C(M, \gamma) \tau_r^{-\gamma}}{\mathbf{K}_0(3C_y(\delta_r^B) - C_0(\delta_r^B))}.$$

Proof of the Claim: As the proof will be concluded by a continuity argument in r , we assume that $\Pi(u, r) \geq K(M)$ and $\delta^B(u(s\mathbf{x})) \leq c(M)$. From (11.1), (4.1) and

Theorem 4.7 we infer that in B_1 , up to a rotation,

$$\begin{aligned}
\Pi(u, r/2) &= \tau_r p_{\delta_r^B} + \Pi(Z_{p_{\delta_r^B}}, 1/2) + O(\tau_r^{-\gamma}) \\
(11.2) \quad &= \left(\tau_r(1 - \delta_r^B) + \mathbf{K}_0((3B_x(1/2 - \delta) - B(1/2 - \delta))x^2 + (3B_y(1/2 - \delta) \right. \\
&\quad \left. - B(1/2 - \delta))y^2 + (3B_z(1/2 - \delta) - B(1/2 - \delta))z^2 \right) + O(\tau_r^{-\gamma}) \\
&= (\tau_r(1 - \delta_r^B) + \mathbf{K}_0(1 - C_0(\delta_r^B)\delta_r^B + o(\delta_r^B)))x^2 \\
&\quad + (\tau_r\delta_r^B + \mathbf{K}_0(3C_y(\delta_r^B)\delta_r^B - C_0(\delta_r^B)\delta_r^B))y^2 \\
&\quad + (-\tau_r + \mathbf{K}_0(-3C_y(\delta_r^B)\delta_r^B + 2C_0(\delta_r^B)\delta_r^B - 1 - o(\delta_r^B)))z^2 + O(\tau_r^{-\gamma}).
\end{aligned}$$

Rotating the coordinate system slightly to $\tilde{x}, \tilde{y}, \tilde{z}$, we deduce from (11.2) that the quotient of the \tilde{y}^2 and the \tilde{x}^2 coefficient of $\Pi(u, r/2)$ is estimated from below by

$$\frac{\delta_{r/2}^B}{1 - \delta_{r/2}^B} \geq \frac{\tau_r\delta_r^B + 3\mathbf{K}_0C_y(\delta_r^B)\delta_r^B - \mathbf{K}_0C_0(\delta_r^B)\delta_r^B - C_1\tau_r^{-\gamma}}{\tau_r(1 - \delta_r^B) + \mathbf{K}_0 - \mathbf{K}_0(C_0(\delta_r^B)\delta_r^B + o(\delta_r^B)) + C_1\tau_r^{-\gamma}}.$$

We maintain that for $\delta_r^B \leq c_2(M, \gamma)$ and $\tau_r \geq C_3(M, \gamma)$,

$$(11.3) \quad \frac{\delta_{r/2}^B}{1 - \delta_{r/2}^B} \geq \frac{\delta_r^B}{1 - \delta_r^B} \text{ unless } \delta_r^B \leq \frac{2C_1\tau_r^{-\gamma}}{\mathbf{K}_0(3C_y(\delta_r^B) - C_0(\delta_r^B))} :$$

Subtracting the two quotients we end up with

$$\begin{aligned}
D &:= \frac{\delta_{r/2}^B}{1 - \delta_{r/2}^B} - \frac{\delta_r^B}{1 - \delta_r^B} \\
&\geq S \left[\mathbf{K}_0(3C_y(\delta_r^B) - C_0(\delta_r^B))\delta_r^B \right. \\
&\quad \left. - \mathbf{K}_0(3C_y(\delta_r^B) - 2C_0(\delta_r^B))(\delta_r^B)^2 - \mathbf{K}_0\delta_r^B + \delta_r^B o(\delta_r^B) - C\tau_r^{-\gamma} \right],
\end{aligned}$$

where

$$S = ((\tau_r(1 - \delta_r^B) + \mathbf{K}_0 - \mathbf{K}_0(C_0(\delta_r^B)\delta_r^B + o(\delta_r^B)) + C\tau_r^{-\gamma})(1 - \delta_r^B))^{-1}.$$

For D to be non-negative $-\delta_r^B$ being by assumption small, τ_r being large and $C_y(\delta_r^B) \approx C_0(\delta_r^B) \approx |\log \delta_r^B|$ by Theorem 4.7—it is sufficient that

$$(11.4) \quad \delta_r^B \geq \frac{2C_1\tau_r^{-\gamma}}{\mathbf{K}_0(3C_y(\delta_r^B) - C_0(\delta_r^B))}.$$

Thus (11.3) holds, and

$$\frac{2C_1\tau_r^{-\gamma}}{\mathbf{K}_0(3C_y(\delta_r^B) - C_0(\delta_r^B))} \leq \delta_r^B \leq c_2(M, \gamma) \text{ implies that } \frac{\delta_{r/2}^B}{1 - \delta_{r/2}^B} \geq \frac{\delta_r^B}{1 - \delta_r^B},$$

which in turn implies that $\delta_{r/2}^B > \delta_r^B$. Using Theorem 4.5 (i) together with Theorem 4.7 (i) while observing that $\frac{\partial A}{\partial \delta_r^B}(\delta_r^B) = -\frac{\partial B}{\partial \delta_r^B}(\frac{1}{2} - \delta_r^B)$ and that $\frac{\partial A_y}{\partial \delta_r^B}(\delta_r^B) = -\frac{\partial B_y}{\partial \delta_r^B}(\frac{1}{2} - \delta_r^B)$, we conclude that

$$(3C_y(\delta_{r/2}^B) - C_0(\delta_{r/2}^B))\delta_{r/2}^B \geq (3C_y(\delta_r^B) - C_0(\delta_r^B))\delta_r^B.$$

As $\tau_{r/2}^{-\gamma} \leq \tau_r^{-\gamma}$, it follows then that

$$(11.5) \quad \delta_{r/2}^B \geq \frac{2C_1\tau_{r/2}^{-\gamma}}{\mathbf{K}_0(3C_y(\delta_r^B) - C_0(\delta_r^B))}.$$

In this case $\delta_{2^{-k}r}^B \geq \delta_r^B$ for $k = 1, 2, \dots$. Altogether we obtain that

$$\text{either } \liminf_{k \rightarrow \infty} \delta_{2^{-k}r}^B > 0 \text{ or } \delta_r^B \leq \frac{2C_1\tau_r^{-\gamma}}{\mathbf{K}_0(3C_y(\delta_r^B) - C_0(\delta_r^B))}.$$

Since $\liminf_{k \rightarrow \infty} \delta_{2^{-k}r}^B > 0$ would contradict our assumption that

$$\Pi(u, r_j) / \sup_{B_1} |\Pi(u, r_j)| \rightarrow x^2 - z^2 \text{ as } j \rightarrow \infty,$$

we have proved the Claim.

In the last part of our proof we will use the decay estimate in the Claim in order to estimate how much $\Pi(u(r \cdot))$ moves when varying r in the interval $(0, s)$. First note that the Claim and the fact that $C_y \approx C_0$ when $\delta \ll 1$ imply that

$$(11.6) \quad C_y(\delta_r^B)\delta_r^B \leq C_4\tau_r^{-\gamma}.$$

Next observe that by (11.2) and (11.6),

$$(11.7) \quad \tau_{r/2} = \tau_r + \mathbf{K}_0 + O(C_y(\delta_r^B)\delta_r^B) = \tau_r + \mathbf{K}_0 + O(\tau_r^{-\gamma}).$$

Using (11.2) once more along with (11.7) and (11.6) we obtain

$$\begin{aligned} & \sup_{B_1} \left| \frac{\Pi(u, r)}{\sup_{B_1} |\Pi(u, r)|} - \frac{\Pi(u, r/2)}{\sup_{B_1} |\Pi(u, r/2)|} \right| \\ & \leq \sup_{B_1} \left| \frac{\tau_r(1 - \delta_r^B)x^2 + \delta_r^B y^2 - z^2}{\tau_r} \right. \\ & \quad - \frac{(\tau_r(1 - \delta_r^B) + \mathbf{K}_0 - \mathbf{K}_0(C_0(\delta_r^B)\delta_r^B + o(\delta_r^B)) + O(\tau_r^{-\gamma}))x^2}{\tau_r + \mathbf{K}_0 + O(\tau_r^{-\gamma})} \\ & \quad - \frac{(\tau_r\delta_r^B + 3\mathbf{K}_0C_y(\delta_r^B)\delta_r^B - \mathbf{K}_0C_0(\delta_r^B)\delta_r^B + O(\tau_r^{-\gamma}))y^2}{\tau_r + \mathbf{K}_0 + O(\tau_r^{-\gamma})} \\ & \quad \left. - \frac{(-\tau_r + \mathbf{K}_0(-3C_y(\delta_r^B)\delta_r^B + 2C_0(\delta_r^B)\delta_r^B - 1 - o(\delta_r^B)))z^2}{\tau_r + \mathbf{K}_0 + O(\tau_r^{-\gamma})} \right| \leq C_5\tau_r^{-(1+\gamma)}. \end{aligned}$$

As in the proof of Theorem 9.1, an iteration leads to

$$\left| \frac{\Pi(u, 2^{-k}s)}{\sup_{B_1} |\Pi(u, 2^{-k}s)|} - (x^2 - z^2) \right| \leq C_6\tau_{2^{-k}s}^{-\gamma},$$

and we obtain the desired estimate as well as

$$\lim_{r \rightarrow 0} \left(\frac{u(\mathbf{x}^0 + r\mathbf{x})}{r^2} - \Pi(u, r, \mathbf{x}^0)(\mathbf{x}) \right) = Z(Q_{\mathbf{x}^0}\mathbf{x}).$$

□

12. STRUCTURE OF THE SINGULAR SET IN \mathbb{R}^3

So far we have shown that if $\Delta u = -\chi_{\{u>0\}}$ in $B_1 \subset \mathbb{R}^3$ then the singular set $S^u = \{x \in B_1 : u(\mathbf{x}) = |\nabla u(\mathbf{x})| = 0 \text{ and } \lim_{r \rightarrow 0} \Phi_{\mathbf{x}}^u(r) = -\infty\}$ is divided into two parts $S_1^u = \{\mathbf{x} \in S^u : \lim_{r \rightarrow 0} \frac{u(rQ\mathbf{x} + \mathbf{x}^0)}{\sup_{B_r} |u|} = \frac{x^2 + y^2}{2} - z^2 \text{ for some } Q \in \mathcal{Q}\} \cup \{\mathbf{x} \in S^u : \lim_{r \rightarrow 0} \frac{u(rQ\mathbf{x} + \mathbf{x}^0)}{\sup_{B_r} |u|} = -(\frac{x^2 + y^2}{2} - z^2) \text{ for some } Q \in \mathcal{Q}\}$ and $S_2^u = \{\mathbf{x} \in S^u : \lim_{r \rightarrow 0} \frac{u(rQ\mathbf{x} + \mathbf{x}^0)}{\sup_{B_r} |u|} = xz \text{ for some } Q \in \mathcal{Q}\}$. In this section we show that S_1^u consists only of isolated points, and that S_2^u is locally contained in a C^1 -curve. We also derive a compactness result for S_2^u .

Lemma 12.1. *Let $n = 3$, let u solve (1.1) and let $\mathbf{x}^0 \in S_1^u$. Then there exists an $r = r(u, \mathbf{x}^0) > 0$ such that $\{u = 0\} \cap \{\nabla u = 0\} \cap B_r(\mathbf{x}^0) = \{\mathbf{x}^0\}$, that is \mathbf{x}^0 is the only singular point in a small neighbourhood of \mathbf{x}^0 . For each class of solutions v sufficiently close to u in $L^\infty(B_1)$, $\{v = 0\} \cap \{\nabla v = 0\}$ contains at most one point in B_r .*

Proof. Suppose towards a contradiction that there exists a sequence of solutions $u^j \rightarrow u$ in $L^\infty(B_1)$ as well as sequences $\{u^j = 0\} \cap \{\nabla u^j = 0\} \ni \mathbf{x}^j \rightarrow \mathbf{x}^0$ and $\{u^j = 0\} \cap \{\nabla u^j = 0\} \setminus \{\mathbf{x}^j\} \ni \mathbf{y}^j \rightarrow \mathbf{x}^0$ as $j \rightarrow \infty$. Let $r_j = |\mathbf{y}^j - \mathbf{x}^j|$. Then, passing if necessary to a subsequence, $(\mathbf{y}^j - \mathbf{x}^j)/r_j \rightarrow \boldsymbol{\xi} \in \partial B_1$. On the other hand, by $W^{2,p}$ -regularity of the solution, $u^j \rightarrow u$ in $L^\infty(B_1) \cap W^{2,p}(B_{1/2})$ so that the assumptions in Theorem 9.1 are satisfied in $B_\rho(\mathbf{x}^j)$ for small ρ and sufficiently large j . Rotating each solution suitably around the origin, we obtain that $\mathbf{x}^j \in S_1^{u^j}$ and that

$$(12.1) \quad \lim_{j \rightarrow \infty} \frac{u^j(r_j \mathbf{x} + \mathbf{x}^j)}{\sup_{B_{r_j}} |u^j|} = w(\mathbf{x}) = \pm \left(\frac{x^2 + y^2}{2} - z^2 \right).$$

By $C^{1,\alpha}$ -convergence in equation (12.1) it follows that $w(\boldsymbol{\xi}) = |\nabla w(\boldsymbol{\xi})| = 0$ which is a contradiction since $|\boldsymbol{\xi}| = 1$ and the origin is the only point where $w = |\nabla w| = 0$. \square

We continue this section with a regularity result for S_2^u .

Theorem 12.2. *Let $n = 3$. If $0 \in S_2^u$ then there exists an $r(u) > 0$ such that $S_2^u \cap B_{r(u)}$ is contained in a C^1 -curve. For each class of solutions v sufficiently close to u in $L^\infty(B_1)$, the curves containing S_2^v are relatively compact in $C^1(B_{r(u)})$.*

Proof. Let us consider a sequence of solutions $u^j \rightarrow u$ in $L^\infty(B_1)$. By uniform $W^{2,p}$ -regularity of the solution, for sufficiently small $s > 0$

$$\Pi(u, s) \geq 2K(M) \text{ and } \delta^B(u(s\mathbf{x})) \leq c(M)/2,$$

and for all sufficiently small $|\mathbf{x}^0|$ and all sufficiently large j ,

$$\Pi(u^j, s, \mathbf{x}^0) \geq K(M) \text{ and } \delta^B(u(\mathbf{x}^0 + s\mathbf{x})) \leq c(M).$$

From Theorem 11.1 we obtain therefore that

$$\sup_{B_1} \left| \frac{u^j(\mathbf{y} + r\mathbf{x})}{\sup_{B_1} |u^j(\mathbf{y} + r\cdot)|} - p_{\mathbf{y}}^j \right| \leq \epsilon_1$$

for all sufficiently large j , all $\mathbf{y} \in S_2^{u^j} \cap B_\rho$, a rotation $Q_{\mathbf{y}}^j$, $p(x_1, x_2, x_3) = 2x_1x_3$, $p_{\mathbf{y}}^j(\mathbf{x}) = p(Q_{\mathbf{y}}^j \mathbf{x})$ and all $r \in (0, r_1)$.

Uniform cone flatness: For each $\epsilon > 0$ there exists an $s_\epsilon > 0$ such that for

sufficiently large j and all $\mathbf{y} \in S_2^{u^j} \cap B_{\rho_1}$, $\{u^j = 0\} \cap \{|\nabla u^j| = 0\} \cap B_s(\mathbf{y}) \subset \{\mathbf{y} + sQ_{\mathbf{y}}(x_1, x_2, x_3) : x_1^2 + x_3^2 \leq \epsilon x_2^2\}$ for $s \in (0, s_\epsilon)$.

Proof of uniform cone flatness: Suppose towards a contradiction that there exists an $\epsilon_0 > 0$, a subsequence of solutions

$$v^j = \frac{u^j(\mathbf{y}^j + s_j Q_{\mathbf{y}^j}^{-1} \cdot)}{\sup_{B_1} |u^j(\mathbf{y}^j + s_j \cdot)|} \rightarrow 2x_1 x_3 \text{ in } C^{1,\alpha}(\overline{B_1})$$

and a sequence of points $\xi^j \rightarrow \xi^0 \in \partial B_1$ such that $v^j(\xi^j) = |\nabla v^j(\xi^j)| = 0$ and $(\xi_1^j)^2 + (\xi_3^j)^2 \geq \epsilon_0$. Then $(\xi_1^0)^2 + (\xi_3^0)^2 \geq \epsilon_0$, contradicting $0 = |\nabla v^0(\xi^0)| = 2\sqrt{(\xi_1^0)^2 + (\xi_3^0)^2}$ and thereby proving uniform cone flatness.

A standard consequence of the uniform cone flatness is that the class of curves containing $S_2^{u^j} \cap B_{\rho_2}$ is for large j relatively compact in C^1 . An argument by contradiction yields the Theorem. \square

The following corollary can be regarded as an extension of [15, Corollary 7.2] outside a small cone (even outside a cusp) in the y -direction.

Corollary 12.3. *Let $n = 3$ and suppose that for some solution u to equation (1.1),*

$$\lim_{j \rightarrow \infty} \frac{u(r_j \mathbf{x})}{\sup_{B_{r_j}} |u|} = 2xz.$$

Then for each $\theta > 0$ there exists an $r(u) > 0$ such that $\{u = 0\} \cap B_{r(u)} \cap \{|y|^2 < \theta(|x|^2 + |z|^2)\}$ consists of two 2-dimensional C^1 -manifolds restricted to $B_{r(u)} \cap \{|y|^2 < \theta(|x|^2 + |z|^2)\}$, intersecting at right angles at the origin in the xz -plane. For each class of solutions v sufficiently close to u in $L^\infty(B_1)$ and having each an S_2^v -point sufficiently close to 0, the manifolds are relatively compact in C^1 .

Proof. The proof is similar to the proof of Corollary 9.2 and left to the reader. \square

13. APPENDIX

Proof of Theorem 4.5:

Let us begin by proving (vi), that is

$$(13.1) \quad 3(A_y''(\delta) - A_x''(\delta)) + 2\delta(3A_y''(\delta) + 3A_x''(\delta) - 2A''(\delta)) + 2(3A_y'(\delta) + 3A_x'(\delta) - 2A'(\delta)) > 0$$

for $\delta \in (0, 1/2)$

For $\delta > 0$ we have that $|\nabla p_\delta| \neq 0$ on $\{p_\delta = 0\} \cap \partial B_1$ and we may thus differentiate A_x , A_y and A . Thus

$$\begin{aligned}
A'_x(\delta) &= 8 \int_0^{\pi/2} \sin^3(\operatorname{arccot}(\sqrt{1/2 + \delta \cos(2\phi)})) \cos^2(\phi) \frac{\partial \operatorname{arccot}(\sqrt{1/2 + \delta \cos(2\phi)})}{\partial \delta} d\phi \\
&= 8 \int_0^{\pi/2} \frac{\partial}{\partial \delta} \left(\frac{1}{12} (\cos(3 \operatorname{arccot}(\sqrt{1/2 + \delta \cos(2\phi)}))) \right. \\
&\quad \left. - 9 \cos(\operatorname{arccot}(\sqrt{1/2 + \delta \cos(2\phi)})) \right) \cos^2(\phi) d\phi \\
&= -\frac{32}{3} \int_0^{\pi/2} \frac{\partial}{\partial \delta} \left(\frac{(2 + \delta \cos(2\phi)) \sqrt{1 + 2\delta \cos(2\phi)}}{(3 + 2\delta \cos(2\phi))^{3/2}} \right) \cos^2(\phi) d\phi, \\
A'_y(\delta) &= 8 \int_0^{\pi/2} \frac{\partial}{\partial \delta} \left(\frac{1}{12} (\cos(3 \operatorname{arccot}(\sqrt{1/2 + \delta \cos(2\phi)}))) \right. \\
&\quad \left. - 9 \cos(\operatorname{arccot}(\sqrt{1/2 + \delta \cos(2\phi)})) \right) \sin^2(\phi) d\phi \\
&= -\frac{32}{3} \int_0^{\pi/2} \frac{\partial}{\partial \delta} \left(\frac{(2 + \delta \cos(2\phi)) \sqrt{1 + 2\delta \cos(2\phi)}}{(3 + 2\delta \cos(2\phi))^{3/2}} \right) \sin^2(\phi) d\phi
\end{aligned}$$

and

$$A'(\delta) = -8 \int_0^{\pi/2} \frac{\partial}{\partial \delta} (\cos(\operatorname{arccot}(\sqrt{1/2 + \delta \cos(2\phi)}))) d\phi.$$

Differentiating once more,

$$\begin{aligned}
A''_x(\delta) &= -\frac{32}{3} \int_0^{\pi/2} \frac{\partial^2}{\partial \delta^2} \left(\frac{(2 + \delta \cos(2\phi)) \sqrt{1 + 2\delta \cos(2\phi)}}{(3 + 2\delta \cos(2\phi))^{3/2}} \right) \cos^2(\phi) d\phi, \\
A''_y(\delta) &= -\frac{32}{3} \int_0^{\pi/2} \frac{\partial^2}{\partial \delta^2} \left(\frac{(2 + \delta \cos(2\phi)) \sqrt{1 + 2\delta \cos(2\phi)}}{(3 + 2\delta \cos(2\phi))^{3/2}} \right) \sin^2(\phi) d\phi
\end{aligned}$$

and

$$A''(\delta) = -8 \int_0^{\pi/2} \frac{\partial^2}{\partial \delta^2} (\cos(\operatorname{arccot}(\sqrt{1/2 + \delta \cos(2\phi)}))) d\phi.$$

Using the last three identities we may write the left hand side in equation (13.1) as the sum of the following three terms (13.2), (13.3) and (13.4):

$$(13.2) \quad 3(A''_y - A''_x) = 32 \int_0^{\pi/2} \frac{\partial^2}{\partial \delta^2} \left(\frac{(2 + \delta \cos(2\phi)) \sqrt{1 + 2\delta \cos(2\phi)}}{(3 + 2\delta \cos(2\phi))^{3/2}} \right) \cos(2\phi) d\phi,$$

$$(13.3) \quad 2\delta(3A''_x + 3A''_y - 2A'') = -32\delta \int_0^{\pi/2} \frac{\partial^2}{\partial \delta^2} \left(\frac{\sqrt{1 + 2\delta \cos(2\phi)}}{(3 + 2\delta \cos(2\phi))^{3/2}} \right) d\phi$$

and

$$(13.4) \quad 2(3A'_x + 3A'_y - 2A') = -32 \int_0^{\pi/2} \frac{\partial}{\partial \delta} \left(\frac{\sqrt{1 + 2\delta \cos(2\phi)}}{(3 + 2\delta \cos(2\phi))^{3/2}} \right) d\phi.$$

In order to estimate (13.2), (13.3) and (13.4), we will use the change of variables $x = \delta \cos(2\phi)$ and then use Taylor expansions: First we notice that —using double factorials $(2k+1)!! = 3 \times 5 \times \cdots \times (2k-1) \times (2k+1)$ —

$$\sqrt{1+2x} = 1 + x + \sum_{k=2}^{\infty} (-1)^{k+1} \frac{(2k-3)!!}{k!} x^k$$

and

$$\frac{1}{(3+2x)^{3/2}} = \frac{1}{3\sqrt{3}} \sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)!!}{3^k k!} x^k,$$

both sums are absolutely convergent for $|x| < 1/2$. Thus

$$\begin{aligned} \frac{\sqrt{1+2x}}{(3+2x)^{3/2}} &= \frac{1+x}{3\sqrt{3}} \left(\sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)!!}{3^k k!} x^k \right) \\ &\quad - \left(\sum_{k=2}^{\infty} (-1)^k \frac{(2k-3)!!}{k!} x^k \right) \left(\frac{1}{3\sqrt{3}} \sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)!!}{3^k k!} x^k \right) \\ &= \frac{1}{3\sqrt{3}} + \frac{1}{3\sqrt{3}} \sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{(2k-1)!!}{3^k k!} (k-1) \right) x^k \\ &\quad - \left(\sum_{k=2}^{\infty} (-1)^k \frac{(2k-3)!!}{k!} x^k \right) \left(\frac{1}{3\sqrt{3}} \sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)!!}{3^k k!} x^k \right). \end{aligned}$$

Notice that the product of the last two sums will equal a sum $\sum_{k=0}^{\infty} a_k x^k$, where $a_k \geq 0$ when k is even and $a_k \leq 0$ when k is odd. Inserting $x = \delta \cos(2\phi)$ and these Taylor expansions in equation (13.3) and using that $a_k \geq 0$ for even k and that $\int_0^{\pi/2} a_k \cos^k(2\phi) d\phi = 0$ for odd k , we see that

$$\begin{aligned} &2\delta(3A''_x + 3A''_y - 2A'') \\ &= \frac{32}{3\sqrt{3}} \delta \int_0^{\pi/2} \frac{\partial^2}{\partial \delta^2} \left(\sum_{k=2}^{\infty} (-1)^k \left(\frac{(2k-1)!!}{3^k k!} (k-1) \right) \delta^k \cos^k(2\phi) \right) d\phi \\ &\quad + \frac{32}{3\sqrt{3}} \delta \int_0^{\pi/2} \frac{\partial^2}{\partial \delta^2} \left(\sum_{k=0}^{\infty} a_k \delta^k \cos^k(2\phi) \right) d\phi > 0. \end{aligned}$$

Similarly we may estimate the left-hand side in (13.4) and obtain that

$$2(3A'_x + 3A'_y - 2A') > 0.$$

Next, we make a Taylor expansion of the integrand in equation (13.2). First, we calculate

$$\begin{aligned} (2+x)\sqrt{1+2x} &= (2+x) \left(1 + x + \sum_{k=2}^{\infty} (-1)^{k+1} \frac{(2k-3)!!}{k!} x^k \right) \\ &= 2 + 3x + \sum_{k=3}^{\infty} (-1)^{k+1} \left(\frac{(2k-5)!!}{k!} (3k-6) \right) x^k. \end{aligned}$$

Therefore

$$\begin{aligned}
& \frac{(2+x)\sqrt{1+2x}}{(3+2x)^{3/2}} \\
&= \frac{1}{3\sqrt{3}}(2+3x) \left(\sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)!!}{3^k k!} x^k \right) \\
&\quad - \left(\sum_{k=3}^{\infty} (-1)^k \frac{(2k-5)!!}{k!} (3k-6)x^k \right) \left(\frac{1}{3\sqrt{3}} \sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)!!}{3^k k!} x^k \right) \\
&= \frac{1}{3\sqrt{3}} \left(2+x + \sum_{k=2}^{\infty} (-1)^{k+1} \left(\frac{(2k-3)!!}{3^k k!} (10k^2 - 9k + 2) \right) x^k \right) \\
&\quad - \left(\sum_{k=3}^{\infty} (-1)^k \frac{(2k-5)!!}{k!} (3k-6)x^k \right) \left(\frac{1}{3\sqrt{3}} \sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)!!}{3^k k!} x^k \right).
\end{aligned}$$

As before, we notice that the product of the last two sums may be written as $\sum_{k=3}^{\infty} a_k x^k$ where $a_k \geq 0$ when k is even and $a_k \leq 0$ when k is odd. Using this together with the above Taylor expansions, the fact that $\int_0^{\pi/2} a_k \cos^{k+1}(2\phi) d\phi = 0$ for even k and that

$$b_k = \left(\frac{(2k-3)!!}{3^k k!} (4k^2 + k - 2) \right) > 0,$$

we obtain that the left-hand side of (13.2),

$$\begin{aligned}
& 3(A_y'' - A_x'') \\
&= \frac{32}{\sqrt{3}} \int_0^{\pi/2} \frac{\partial^2}{\partial \delta^2} \left(\left(\sum_{k=2}^{\infty} (-1)^{k+1} b_k \delta^k \cos^k(2\phi) \right) \right) \cos(2\phi) d\phi \\
&\quad - 32 \int_0^{\pi/2} \frac{\partial^2}{\partial \delta^2} \left(\left(\sum_{k=2}^{\infty} a_k \delta^k \cos^{k+1}(2\phi) \right) \right) d\phi > 0.
\end{aligned}$$

Therefore

$$3(A_y'' - A_x'') + 2\delta(3A_y'' + 3A_x'' - 2A'') + 2(3A_y' + 3A_x' - 2A') > 0$$

and (vi) holds.

Estimate (vii) follows now in a straightforward way: Since $3A_x(\delta) - A(\delta) < 0$ for $\delta \in (0, 1/2)$ it is sufficient to show that

$$F(\delta) := (1+2\delta)(3A_y(\delta) - A(\delta)) - (1-2\delta)(3A_x(\delta) - A(\delta)) < 0.$$

By symmetry we have $F(0) = 0$. Moreover, rotating $\Pi(Z, 1/2) = (\log(2)/\pi)xz$ (Lemma 4.3) in the xz -plane by 45° , we obtain that $3A_y(1/2) - A(1/2) = 0$, proving (v) as well as $F(1/2) = 0$. By (vi) we also know that F is convex:

$$F(\delta) < (1-2\delta)F(0) + 2\delta F(1/2) = 0,$$

and (vii) follows.

Next we prove (i), that is

$$\frac{\partial(3A_y(\delta) - A(\delta))}{\partial \delta} > 0 \text{ for } \delta \in (1, 1/2).$$

First,

$$\begin{aligned} & \frac{\partial(3A_y(\delta) - A(\delta))}{\partial\delta} \\ &= 8 \int_0^{\pi/2} \frac{\partial}{\partial\delta} \left[-\frac{\sqrt{1+2\delta\cos(2\phi)}}{(3+2\delta\cos(2\phi))^{3/2}} + 2\frac{(2+\delta\cos(2\phi))\sqrt{1+2\delta\cos(2\phi)}}{(3+2\delta\cos(2\phi))^{3/2}} \cos(2\phi) \right] d\phi. \end{aligned}$$

As before we may write the integrand as

$$\frac{\partial}{\partial\delta} \left[-\frac{\sqrt{1+2x}}{(3+2x)^{3/2}} + 2\frac{(2+x)\sqrt{1+2x}}{(3+2x)^{3/2}} \cos(2\phi) \right],$$

where $x = \delta \cos(2\phi)$. Using the Taylor series expansions calculated before it is easy to see that

$$\begin{aligned} & 8 \int_0^{\pi/2} \frac{\partial}{\partial\delta} \left[-\frac{\sqrt{1+2\delta\cos(2\phi)}}{(3+2\delta\cos(2\phi))^{3/2}} + 2\frac{(2+\delta\cos(2\phi))\sqrt{1+2\delta\cos(2\phi)}}{(3+2\delta\cos(2\phi))^{3/2}} \cos(2\phi) \right] d\phi \\ &= \int_0^{\pi/2} \left[\sum_{k=0}^{\infty} (-1)^k a_k \delta^{k-1} \cos^k(2\phi) + \sum_{k=0}^{\infty} (-1)^{k+1} b_k \delta^{k-1} \cos^{k+1}(2\phi) \right] d\phi, \end{aligned}$$

where $a_k, b_k \geq 0$. Using $\int_0^{\pi/2} \cos^j(2\phi) d\phi = 0$ for odd j implies that

$$\frac{\partial(3A_y(\delta) - A(\delta))}{\partial\delta} > 0.$$

We argue similarly to show (ii), that is

$$\frac{\partial(3A_x(\delta) - A(\delta))}{\partial\delta} < 0.$$

Here

$$\begin{aligned} & \frac{\partial(3A_x(\delta) - A(\delta))}{\partial\delta} \\ &= -8 \int_0^{\pi/2} \frac{\partial}{\partial\delta} \left[\frac{(1+4\cos(2\phi) + 2\cos^2(2\phi)\delta)\sqrt{1+2\delta\cos(2\phi)}}{(3+2\delta\cos(2\phi))^{3/2}} \right] d\phi \\ (13.5) \quad &= -8 \int_0^{\pi/2} \frac{2\cos^2(2\phi)(3-2\delta)}{\sqrt{1+2\delta\cos(2\phi)}(3+2\delta\cos(2\phi))^{5/2}} d\phi. \end{aligned}$$

Substituting the Taylor expansions

$$\frac{1}{\sqrt{1+2\delta\cos(2\phi)}} = \sum_{k=0}^{\infty} (-1)^k \frac{(2k-1)!!}{k!} x^k$$

and

$$\frac{1}{(3+2\cos(2\phi))^{5/2}} = \frac{1}{3\sqrt{3}} \sum_{k=0}^{\infty} (-1)^k \frac{(2k+3)!!}{3^k k!} x^k$$

into (13.5) we may deduce that

$$\begin{aligned} & \frac{\partial(3A_x(\delta) - A(\delta))}{\partial\delta} \\ &= -8 \int_0^{\pi/2} \frac{2 \cos^2(2\phi)(3 - 2\delta)}{3\sqrt{3}} \left[\left(\sum_{k=0}^{\infty} (-1)^k \frac{(2k-1)!!}{k!} \delta^k \cos^k(2\phi) \right) \right. \\ & \quad \left. \times \left(\sum_{k=0}^{\infty} (-1)^k \frac{(2k+3)!!}{3^k k!} \delta^k \cos^k(2\phi) \right) \right] d\phi. \end{aligned}$$

Using that $\delta < 1/2$ and that all the odd terms in the product of the sums equal zero we see that this expression is negative, proving (ii).

From the fact that $A_x(0) = A_y(0)$ (the projection Π preserves symmetry) we infer now that

$$\begin{aligned} 3A_x(0) &= 3/2 A_x(0) + 3/2 A_y(0) = A_x(0) + A_y(0) - \int_{\partial B_1 \cap \{x^2+y^2 > 2z^2\}} (x^2 + y^2)/2 \\ &< A_x(0) + A_y(0) + \int_{\partial B_1 \cap \{x^2+y^2 > 2z^2\}} -z^2 = \int_{\partial B_1 \cap \{x^2+y^2 > 2z^2\}} -1 = A(0), \end{aligned}$$

which proves (iii).

Combining (ii) and (iii) we obtain $3A_x(\delta) - A(\delta) < 0$ for $\delta \in (0, 1/2)$, namely (iv).

Last, we verify (viii)-(ix):

$$\frac{\partial}{\partial\delta}(3A_x(\delta) - A(\delta)) \Big|_{\delta=0} = - \int_0^{\pi/2} \frac{32 \cos^2(\phi) \cos(2\phi)}{3\sqrt{3}} - \frac{16 \cos(2\phi)}{3\sqrt{3}} d\phi = -\frac{4\pi}{3\sqrt{3}}$$

and

$$\frac{\partial}{\partial\delta}(3A_y(\delta) - A(\delta)) \Big|_{\delta=0} = - \int_0^{\pi/2} \frac{32 \sin^2(\phi) \cos(2\phi)}{3\sqrt{3}} - \frac{16 \cos(2\phi)}{3\sqrt{3}} d\phi = \frac{4\pi}{3\sqrt{3}}.$$

□

Proof of Lemma 4.6:

From Theorem 4.5 we deduce that

$$\begin{aligned} 3A_x(1/2) - A(1/2) &< 0, \\ 3A_y(1/2) - A(1/2) &= 0, \text{ and} \\ 3A_x(0) - A(0) &= 3A_y(0) - A(0) < 0. \end{aligned}$$

Observe now that we have proved in (13.3) that

$$3A_x'' + 3A_y'' - 2A'' = -16 \int_0^{\pi/2} \frac{\partial^2}{\partial\delta^2} \left(\frac{\sqrt{1 + 2\delta \cos(2\phi)}}{(3 + 2\delta \cos(2\phi))^{3/2}} \right) d\phi > 0 \text{ for } \delta \in (0, 1/2).$$

It follows that

$$\begin{aligned} -(3A_x(\delta) - A(\delta)) - (3A_y(\delta) - A(\delta)) &\geq \inf_{\delta \in (0, 1/2)} [-(3A_x(\delta) - A(\delta)) - (3A_y(\delta) - A(\delta))] \\ &=: d_0 > 0 \text{ for } \delta \in [0, 1/2]. \text{ Now by (4.1),} \end{aligned}$$

$$\Pi(Z_{p_\delta}, 1/2) = -\mathbf{K}_0((3A_x(\delta) - A(\delta))x^2 + (3A_y(\delta) - A(\delta))y^2 + (3A_z(\delta) - A(\delta))z^2).$$

As $\Pi(Z_{p_\delta}, 1/2)$ is harmonic and thus

$$(3A_z(\delta) - A(\delta)) = -((3A_x(\delta) - A(\delta)) + (3A_y(\delta) - A(\delta))) \geq d_0,$$

we obtain that

$$\begin{aligned} \sup_{B_1} |Cp_\delta + \Pi(Z_{p_\delta}, 1/2)| &\geq |Cp_\delta + \Pi(Z_{p_\delta}, 1/2)|(0, 0, 1) \\ &= |-C - \mathbf{K}_0(3A_z(\delta) - A(\delta))| \geq C + \mathbf{K}_0 d_0, \end{aligned}$$

and the Lemma follows with

$$(13.6) \quad \eta_0 = \mathbf{K}_0 d_0 / 2.$$

□

Proof of Theorem 4.7:

We begin by proving (i): For sufficiently small $\delta > 0$ we have

$$\begin{aligned} (13.7) \quad -\frac{\partial B_y(\delta)}{\partial \delta} &= 8 \frac{\partial}{\partial \delta} \int_0^{\pi/2} \int_{\arccot(\sqrt{(1-\delta)\cos^2(\phi)+\delta\sin^2(\phi)})}^{\pi/2} \sin^3(\theta) \sin(\phi)^2 d\theta d\phi \\ &= 4 \int_0^{\pi/2} \frac{\sin^2(\phi)}{(1 + (1-\delta)\cos^2(\phi) + \delta\sin^2(\phi))^{5/2}} \frac{\sin^2(\phi) - \cos^2(\phi)}{\sqrt{(1-\delta)\cos^2(\phi) + \delta\sin^2(\phi)}} d\phi \\ &\geq -C_1 + 4 \int_{\pi/4}^{\pi/2} \frac{\sin^2(\phi)}{(1 + (1-\delta)\cos^2(\phi) + \delta\sin^2(\phi))^{5/2}} \frac{\sin^2(\phi) - \cos^2(\phi)}{\sqrt{(1-\delta)\cos^2(\phi) + \delta\sin^2(\phi)}} d\phi \\ &\geq -C_1 + c_2 \int_{\pi/4}^{\pi/2} \frac{1}{\sqrt{(1-\delta)\cos^2(\phi) + \delta\sin^2(\phi)}} d\phi \\ &= -C_1 + c_2 \int_{\pi/4}^{\pi/2} \frac{1}{\sqrt{\cos^2(\phi) - \delta\cos(2\phi)}} d\phi. \end{aligned}$$

Next we notice that when $\phi \in (3\pi/8, \pi/2)$, $-\cos(2\phi) > 1/\sqrt{2}$. Consequently the right-hand side in estimate (13.7) is estimated from below by

$$\begin{aligned} &-C_1 + c_2 \int_{3\pi/8}^{\pi/2} \frac{1}{\sqrt{\cos^2(\phi) + \delta/\sqrt{2}}} d\phi \\ &\geq -C_1 + c_3 \int_0^{1-\sin(3\pi/8)} \frac{1}{\sqrt{2t-t^2} \sqrt{\delta/\sqrt{2} + 2t-t^2}} dt \\ &\geq -C_1 + c_4 \int_0^{1-\sin(3\pi/8)} \frac{1}{\sqrt{t}\sqrt{c\delta+t}} dt \geq -C_1 + c_5 \log \frac{1}{\sqrt{c\delta}}. \end{aligned}$$

It follows that for sufficiently small $\delta > 0$,

$$-\frac{\partial B_y(\delta)}{\partial \delta} \geq -C_6 + c_7 \log\left(\frac{1}{\delta}\right).$$

In particular,

$$B_y(0) - B_y(\delta) \geq \left(-C_6 + c_7 \log\left(\frac{1}{\delta}\right)\right)\delta.$$

A similar calculation shows that

$$B_y(0) - B_y(\delta) \leq \left(C_8 - C_9 \log\left(\frac{1}{\delta}\right)\right)\delta,$$

so that (i) holds.

Next we are going to prove that

$$\begin{aligned}
 (13.8) \quad & \left| \frac{\partial (B_y(\delta) - B(\delta))}{\partial \delta} \right| \\
 &= \left| 4 \int_0^{\pi/2} \left(\frac{-2 \cos^2(\phi) + \delta \cos(2\phi)}{(1 + (1 - \delta) \cos^2(\phi) + \delta \sin^2(\phi))^{5/2}} \frac{\sin^2(\phi) - \cos^2(\phi)}{\sqrt{(1 - \delta) \cos^2(\phi) + \delta \sin^2(\phi)}} \right) d\phi \right| \\
 &\leq C_{10},
 \end{aligned}$$

which will imply that

$$(13.9) \quad |C_y(\delta) - C_0(\delta)| \leq C_{11}$$

and, when combined with (i), prove (ii) and (iv).

In order to prove the inequality in (13.8), we make the change of variables $\cos(\phi) = t$, implying that

$$\begin{aligned}
 & \left| 4 \int_0^{\pi/2} \left(\frac{-2 \cos^2(\phi) + \delta \cos(2\phi)}{(1 + (1 - \delta) \cos^2(\phi) + \delta \sin^2(\phi))^{5/2}} \frac{\sin^2(\phi) - \cos^2(\phi)}{\sqrt{(1 - \delta) \cos^2(\phi) + \delta \sin^2(\phi)}} \right) d\phi \right| \\
 &= \left| 4 \int_0^1 \frac{-2t^2 + 2\delta t^2 - \delta}{(1 + t^2 - 2\delta t^2 + \delta)^{5/2}} \frac{1 - 2t^2}{\sqrt{t^2 - 2\delta t^2 + \delta}} \frac{1}{\sqrt{1 - t^2}} dt \right| \\
 &\leq \left| -4C_{12} \int_0^1 \frac{(\delta - 2(1 - \delta)t^2)(1 - 2t^2)}{\sqrt{1 + t}\sqrt{1 - t}\sqrt{\delta + t^2 - 2\delta t^2}} dt \right|.
 \end{aligned}$$

At $t = 1$ the singularity is of order $\frac{1}{\sqrt{1-t}}$ which is integrable. We may thus estimate

$$\begin{aligned}
 & \left| \frac{\partial (B_y(\delta) - B(\delta))}{\partial \delta} \right| \\
 &\leq C_{13} \left(1 + \left| \int_0^{1/2} \frac{\delta - 2(1 - \delta)t^2}{\sqrt{\delta + (1 - 2\delta)t^2}} dt \right| \right) \leq C_{14} \left(1 + \sqrt{\delta} \right) \\
 &+ \left| \int_0^{1/2} \frac{-2(1 - \delta)t^2}{\sqrt{\delta + (1 - 2\delta)t^2}} dt \right| \leq C_{15} (1 + \left| \int_0^{1/2} t \right|) \leq C_{14}.
 \end{aligned}$$

Last, we are going to show (iii), i.e.

$$B_x(\delta) = B_x(0) + o(\delta).$$

First we notice that for $\delta > 0$, $|\nabla p_\delta| \neq 0$ on the set $\{p_\delta = 0\}$ so that we may differentiate $B_x(\delta)$:

$$\begin{aligned}
 (13.10) \quad & \frac{\partial B_x(\delta)}{\partial \delta} = 4 \int_0^{\pi/2} \frac{\cos(2\phi) \cos^2(\phi)}{\sqrt{(1 - \delta) \cos^2(\phi) + \delta \sin^2(\phi)} (1 + (1 - \delta) \cos^2(\phi) + \delta \sin^2(\phi))^{5/2}} d\phi \\
 &= 4 \int_0^{\pi/2} \left(\frac{\cos(2\phi)}{(1 + (1 - \delta) \cos^2(\phi) + \delta \sin^2(\phi))^{5/2}} \right) \left(\frac{\cos^2(\phi)}{\sqrt{(1 - \delta) \cos^2(\phi) + \delta \sin^2(\phi)}} \right) d\phi.
 \end{aligned}$$

The term inside the first parenthesis is smooth for all δ and thus harmless. The term inside the second parenthesis can be estimated by

$$\frac{\cos^2(\phi)}{\sqrt{(1-\delta)\cos^2(\phi) + \delta\sin^2(\phi)}} \leq \frac{|\cos(\phi)|}{\sqrt{1-\delta}},$$

which is bounded for $\delta \in (0, 1/2)$. Using the primitive function

$$\int \frac{\cos(2\phi)\cos(\phi)}{(1+\cos(2\phi))^{5/2}} d\phi = -\frac{1}{2} \frac{\sqrt{1-\cos^2(\phi)}\cos^2(\phi)}{(1+\cos^2(\phi))^{3/2}} + C,$$

we obtain

$$\begin{aligned} B_x(\delta) &= B_x(0) + \delta B'_x(0) + o(\delta) \\ &= B_x(0) + 4\delta \int_0^{\pi/2} \frac{\cos(2\phi)\cos(\phi)}{(1+\cos(2\phi))^{5/2}} d\phi + o(\delta) = B_x(0) + o(\delta). \end{aligned}$$

□

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